

Orbits of curves under the Johnson kernel

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Abstract

This paper has two main goals. First, we give a complete, explicit, and computable solution to the problem of when two simple closed curves on a surface are in the same orbit under the Johnson kernel. Second, we develop an approach to the Johnson filtration and the Johnson homomorphism which is functorial under inclusions of subsurfaces. The key point is that the latter reduces the former to a finite computation, which can be carried out by hand. In particular this solves the conjugacy problem in the Johnson kernel for separating twists. One result of independent interest is the complete description of the restriction to subsurfaces of the lower central series of a surface group.

1 Introduction

This paper has two main goals. First, we give a complete, explicit, and computable solution to the problem of when two simple closed curves on a surface are in the same orbit under the Johnson kernel. Second, we develop an approach to the Johnson filtration and Johnson homomorphism which is functorial under inclusions of subsurfaces. The key point is that the latter reduces the former to a finite computation, which can be carried out by hand.

Mapping class group orbits. Let $\Sigma = S_{g,1}$ be a surface of genus g with one boundary component. The *mapping class group* $\text{Mod}(\Sigma)$ is the group of self-homeomorphisms of Σ fixing $\partial\Sigma$ up to homotopy fixing $\partial\Sigma$. Given a subgroup $\Gamma < \text{Mod}(\Sigma)$, one basic but fundamental question is to determine when two simple closed curves lie in the same orbit under Γ . When $\Gamma = \text{Mod}(\Sigma)$ is the entire mapping class group, the answer follows from the classification of surfaces and was known to Dehn: a complete invariant is the genus of the subsurface cut off by the curve, or 0 if the curve does not separate the surface.

Torelli group orbits. The *Torelli group* $\mathcal{I}(\Sigma)$ is the subgroup of $\text{Mod}(\Sigma)$ consisting of homeomorphisms acting trivially on the homology of Σ . The question of when two simple closed curves lie in the same orbit under the Torelli group is more subtle, and was resolved by Johnson in [5]. One obviously necessary condition for two curves to lie in the same $\mathcal{I}(\Sigma)$ -orbit is that the two curves must be homologous. Johnson proved that for nonseparating curves this is also a sufficient condition. For separating curves this invariant is trivial, since any separating curve is nullhomologous. However, in this case we have another necessary condition: the subspaces of homology cut off by each curve must coincide. Johnson proved that for separating curves, this is again a sufficient condition.

Johnson kernel orbits. The Torelli group is the beginning of a natural filtration on $\text{Mod}(\Sigma)$, and the next term is the *Johnson kernel* $\mathcal{K}(\Sigma)$. Johnson [7] proved that $\mathcal{K}(\Sigma)$ is exactly the subgroup of $\text{Mod}(\Sigma)$ generated by Dehn twists about separating curves. In this paper, we extend Johnson's work to the more difficult question of when two simple closed curves lie in the same orbit under $\mathcal{K}(\Sigma)$.

Orbits of nonseparating curves. We first consider the case when the curves are both nonseparating. Since $\mathcal{K}(\Sigma)$ is contained in $\mathcal{I}(\Sigma)$ we know the curves must be homologous to have any hope of being in the same $\mathcal{K}(\Sigma)$ -orbit, so assume that γ and δ are two nonseparating simple closed curves, both representing the homology class $a \in H_1(\Sigma)$. Note that there is a natural symplectic form ω on $H_1(\Sigma)$ defined by taking $\omega(x, y)$ to be the algebraic intersection number of x and y . Thus we may speak about the perpendicular subspace $a^\perp < H_1(\Sigma)$; since the form is symplectic, note that $a \in a^\perp$.

Theorem 1.1 (Orbits of nonseparating curves). *If γ and δ are nonseparating curves homologous to $a \in H_1(\Sigma)$, there is an explicit, computable, surjective invariant*

$$d_a(\gamma, \delta) \in a^\perp \wedge a^\perp / a \wedge a^\perp$$

defined in Definition 7.4, with the property that:

$$d_a(\gamma, \delta) = 0 \quad \Longleftrightarrow \quad \gamma \text{ and } \delta \text{ lie in the same } \mathcal{K}(\Sigma)\text{-orbit}$$

Orbits of separating curves. We now consider two separating curves. Any separating curve cuts Σ into two subsurfaces, one containing the boundary $\partial\Sigma$ and one not. This induces a splitting

$$H_1(\Sigma) = V \oplus V^\perp$$

where V is the subspace spanned by the subsurface not containing the boundary. The subspaces V and V^\perp are necessarily symplectic subspaces, and orthogonal with respect to the algebraic intersection form. We denote by $\omega_V \in \bigwedge^2 V$ the restriction of the symplectic form ω to V .

Theorem 1.2 (Orbits of separating curves). *If γ and δ are separating curves cutting off the symplectic subspace $V < H_1(\Sigma)$, there is an explicit, computable, surjective invariant*

$$d_V(\gamma, \delta) \in (V \otimes \bigwedge^2 V^\perp) \oplus (V^\perp \otimes \bigwedge^2 V) \quad / \quad (V^\perp \otimes \omega_V)$$

defined in Definition 7.7, with the property that:

$$d_V(\gamma, \delta) = 0 \quad \Longleftrightarrow \quad \gamma \text{ and } \delta \text{ lie in the same } \mathcal{K}(\Sigma)\text{-orbit}$$

From the arguments in Section 7 it will be clear how to extend these results to other configurations of curves; for example, it would be very easy to extend Theorem 1.2 to arbitrary collections of separating curves, or to nonseparating collections of nonseparating curves.

Johnson homomorphism for subsurfaces. All these computations rest on a new machinery for understanding the Johnson homomorphism. The *Johnson homomorphism* $\tau: \mathcal{I}(\Sigma) \rightarrow \bigwedge^3 H_1(\Sigma)$, defined by Johnson in [4], is the most important invariant for elements of the Torelli group. In his thesis, Putman [10] developed the theory of “partitioned surfaces” and used them to define the Torelli group on subsurfaces. The key benefit is that this makes it possible to apply inductive arguments, which have been fundamental in the analysis of mapping class groups for decades, to the Torelli group as well. In this paper we extend Putman’s work by defining for any partitioned surface Σ the *partitioned Johnson homomorphism*

$$\tau_S: \mathcal{I}(\Sigma) \rightarrow W_\Sigma.$$

In Theorem 5.12 we prove that this partitioned Johnson homomorphism is natural under inclusions of subsurfaces. This explains and gives new perspectives on all of Johnson’s computations for

the Johnson homomorphism. In particular, using the partitioned Johnson homomorphism, any homeomorphism which is supported on a smaller subsurface can be analyzed locally on that subsurface. In practice this reduces all the calculations needed for the main theorems to trivial or nearly-trivial computations. The key result necessary for the theorems stated above is a computation of the image of τ_Σ .

Johnson filtration. The technical heart of the paper is a concrete description of the central series induced on a subsurface from the lower central series on the larger surface, and an explicit presentation for the associated graded Lie algebra, carried out in Section 3. This lets us define the entire Johnson filtration on any partitioned surface in Section 4. One surprising corollary is that the restriction of the Johnson filtration to a subsurface only depends on the homological partition associated to that surface, and not on any higher-degree data associated to the embedding.

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2 Background

2.1 Partitioned surfaces

Let S be a compact connected surface with nonempty boundary, with a partition \mathcal{P} of its set of boundary components $\pi_0(\partial S)$, and a basepoint $*$ $\in \partial S$; we call $\Sigma = (S, \mathcal{P}, *)$ a *partitioned surface*. This notion was first used by Putman in [10]. We distinguish the subset $P_0 \in \mathcal{P}$ which contains the component containing $*$.

The metaphor underlying all our terminology regarding partitioned surfaces is that S is embedded into a larger surface S' , and the partition \mathcal{P} records which components can be connected by a path in $S' \setminus \mathring{S}$. We say that two boundary components are *connected outside* Σ if they lie in the same subset $P \in \mathcal{P}$. A separating curve γ on S is called *\mathcal{P} -separating* (or just *separating* if the partition is clear from context) if each subset $P \in \mathcal{P}$ of boundary components lies entirely on one side or the other of γ . We say that a boundary component z is *separating* if $\{z\} \in \mathcal{P}$, and that the partition \mathcal{P} is *totally separated* if each boundary component is separating. If \mathcal{P} contains only a single element (and $|\pi_0(\partial S)| > 1$), we say that Σ is *nonseparating*, since in this case no curve on S which separates any boundary components can be \mathcal{P} -separating. We denote the mapping class group of S by $\text{Mod}(S)$ or $\text{Mod}(\Sigma)$ depending on context. (Note that we require elements of $\text{Mod}(\Sigma)$ to fix each boundary component pointwise.) Throughout this paper, Dehn twists are twists to the *right*.

If S is a subsurface of a surface S' , we say that a path lies *outside* S if it is contained in $S' \setminus \mathring{S}$. If S' is a closed surface, S inherits a partition of its boundary components from S' by defining two components of ∂S to be connected outside Σ if they can be connected by a path outside S . More generally, if S is a subsurface of a partitioned surface Σ' , S inherits the structure of a partitioned surface $\Sigma = (S, \mathcal{P}, *)$ as follows. The partition \mathcal{P} is defined by saying that two components $z_1, z_2 \in \pi_0(\partial S)$ are connected outside Σ if either there is a path outside S from z_1 to z_2 , or there exist components $z'_1, z'_2 \in \pi_0(\partial S')$ with paths outside S from z_i to z'_i and such that z'_1 and z'_2 are connected outside Σ' . For the basepoint $*$ we choose any point in ∂S that can be connected to $*'$ by a path outside S .

2.2 The Torelli group

In this section, we define the homology of a partitioned surface Σ , which we denote by $H(\Sigma)$; this originally appeared in Putman [10] using a different but equivalent definition. Given a partitioned surface Σ , we construct a totally separated surface $\widehat{\Sigma} = (\widehat{S}, \widehat{\mathcal{P}}, \widehat{*})$ with a canonical embedding $\Sigma \rightarrow \widehat{\Sigma}$. For each subset $P \in \mathcal{P}$ with $|P| = n$, we take a surface $S_{0,n+1}$ of genus 0 with $n+1$ boundary components, and glue all but one of these to the n boundary components in P . (Notice that when $n = 1$ this operation is effectively trivial.) The resulting surface \widehat{S} has $|\pi_0(\partial\widehat{S})| = |\mathcal{P}|$; the partition $\widehat{\mathcal{P}}$ is the totally separated partition consisting of singleton sets. We take $\widehat{*} \in \partial\widehat{S}$ to be any point so that $*$ and $\widehat{*}$ lie in the same component of $\widehat{S} \setminus \mathring{S}$. The role of the embedding $\Sigma \rightarrow \widehat{\Sigma}$ is captured by the property that those components of S that are connected outside Σ are exactly those that are connected outside S in \widehat{S} . This embedding is universal, in that any embedding $\Sigma \rightarrow T$ with T totally separated factors through $\Sigma \rightarrow \widehat{\Sigma}$.

The inclusion of $\partial\widehat{S}$ into \widehat{S} gives a map from $H_1(\partial\widehat{S})$ to $H_1(\widehat{S})$, and we denote its image by $H_1(\partial\widehat{S})$. We define

$$H(\Sigma) := H_1(\widehat{S}) / \overline{H_1(\partial\widehat{S})}.$$

Since $\widehat{\mathcal{P}}$ is discrete, the remark above implies that $\widehat{\widehat{\Sigma}} = \widehat{\Sigma}$; it follows that $H(\Sigma) = H(\widehat{\Sigma})$. Extending a homeomorphism φ of Σ by the identity to $\widehat{\Sigma}$ gives an action of φ on $H(\Sigma)$. We define the Torelli group of Σ as

$$\mathcal{I}(\Sigma) := \{\varphi \in \text{Mod}(\Sigma) \mid \varphi \text{ acts trivially on } H(\Sigma)\}.$$

Thus we get an exact sequence

$$1 \rightarrow \mathcal{I}(\Sigma) \rightarrow \text{Mod}(\Sigma) \rightarrow \text{Aut}(H(\Sigma))$$

but the latter map is not in general surjective. We will identify the image of $\text{Mod}(\Sigma)$ in $\text{Aut}(H(\Sigma))$ in Lemma 7.2.

For future reference, we give another definition of $H(\Sigma)$. Given a partitioned surface Σ , we define $\overline{\Sigma}$, a surface with one boundary component, by gluing a disk to each boundary component of $\widehat{\Sigma}$ except the component containing $\widehat{*}$. (This is equivalent to gluing an $S_{0,|P_0|+1}$ to P_0 and an $S_{0,|P|}$ to each other subset $P \in \mathcal{P}$ in ∂S .) The Mayer–Vietoris sequence implies that $H(\Sigma) = H_1(\overline{S})$. The intersection form on $H_1(\overline{S})$ is a $\text{Mod}(\Sigma)$ -invariant symplectic form, and so we conclude that $H(\Sigma)$ is self-dual as a $\text{Mod}(\Sigma)$ -module.

2.3 The Torelli category

Putman defines a category whose objects are partitioned surfaces and whose morphisms are inclusions of subsurfaces respecting the partitions. For our purposes, we will need a refinement of this category, as follows. Given two partitioned surfaces Σ_1 and Σ_2 and an inclusion $i: S_1 \rightarrow S_2$, we say (identifying S_1 with its image $i(S_1)$ when necessary) that:

- *i respects the partitions* if \mathcal{P}_1 -separating and \mathcal{P}_1 -nonseparating curves are taken to \mathcal{P}_2 -separating and \mathcal{P}_2 -nonseparating curves respectively; and
- *i preserves basepoints* if $*_1$ and $*_2$ lie in the same component of $S_2 \setminus \mathring{S}_1$.

For any such map i , S_1 inherits the structure of a partitioned surface from Σ_2 ; an inclusion satisfies these two properties exactly when the inherited structure on S_1 is Σ_1 .

The category \mathcal{TSurf} is defined as follows. Its objects are partitioned surfaces $\Sigma = (S, \mathcal{P}, *)$. A morphism $\iota: \Sigma_1 \rightarrow \Sigma_2$ is an inclusion $i: S_1 \rightarrow S_2$ that respects the partitions and preserves basepoints, together with an inclusion $\hat{i}: \hat{S}_1 \rightarrow \hat{S}_2$ extending i . Note in particular that the canonical inclusion $S \rightarrow \hat{S}$ induces a morphism $\Sigma \rightarrow \hat{\Sigma}$ for any Σ . The inclusion \hat{i} induces a map $H_1(\hat{\Sigma}_1) \rightarrow H_1(\hat{\Sigma}_2)$; this descends to an inclusion $\iota_*: H(\Sigma_1) \rightarrow H(\Sigma_2)$.

If Σ_2 is a partitioned surface, any inclusion $i: S_1 \rightarrow S_2$ gives the subsurface the structure of a partitioned surface Σ_1 . This inclusion always extends to a morphism $\iota: \Sigma_1 \rightarrow \Sigma_2$, but not canonically; the ambiguity is in the choice of the map $\hat{i}: \hat{S}_1 \rightarrow \hat{S}_2$, or equivalently in the choice of the inclusion $\iota_*: H(\Sigma_1) \rightarrow H(\Sigma_2)$.

We highlight two particular types of inclusions:

- ι is *non-collapsing* if for each component T of $S_2 \setminus \mathring{S}_1$ we have $\partial T \not\subset \partial S_1$;
- ι is a *simple capping* if $S_2 \setminus \mathring{S}_1$ is a single disk.

Any inclusion can be factored as the composition of a single non-collapsing inclusion with a sequence of simple cappings; for this reason, we will often consider these special cases separately. Note that since a simple capping respects the partitions, the boundary component which is capped off must be separating.

Given a morphism $\iota: \Sigma_1 \rightarrow \Sigma_2$, extension by the identity induces a map $\text{Mod}(\Sigma_1) \rightarrow \text{Mod}(\Sigma_2)$, which restricts to a map $\iota_*: \mathcal{I}(\Sigma_1) \rightarrow \mathcal{I}(\Sigma_2)$; for non-collapsing inclusions this map is injective. Putman showed in [10] that the Torelli group can be regarded as a functor \mathcal{I} from \mathcal{TSurf} to the category of abelian groups and homomorphisms (compare with Theorem 5.14). Our category \mathcal{TSurf} is actually a refinement of the category considered by Putman; one key benefit of this refinement is that the assignment $\Sigma \rightarrow H(\Sigma)$ becomes functorial.

3 The lower central series on a subsurface

When a subsurface $\Sigma \subset S_0$ is embedded in a closed surface S_0 , the restriction of the lower central series of $\pi_1(S_0)$ gives a filtration of $\pi_1(S)$. In this section we use the subgroup $T(\Sigma)$ to characterize this filtration of $\pi_1(S)$. The main idea of this section is that if the associated graded Lie algebra of a filtration happens to be free, to describe the filtration it suffices to find a free basis. Moreover, if the purported free basis is known to generate, its freeness can be verified by mapping to a Lie algebra already known to be free.

3.1 The restriction to a subsurface of the lower central series

If Γ_j is the lower central series of a group $\Gamma = \Gamma_1$, define $\mathcal{L}_j := \Gamma_j / \Gamma_{j+1}$. Since $[\Gamma_i, \Gamma_j] \subset \Gamma_{i+j}$, the commutator bracket on Γ_1 descends to a bilinear map $\mathcal{L}_i \otimes \mathcal{L}_j \rightarrow \mathcal{L}_{i+j}$. This makes the associated graded $\mathcal{L} := \bigoplus \mathcal{L}_j$ into a Lie algebra. It is well-known that if Γ is free with basis $\{x_1, \dots, x_n\}$, then \mathcal{L} is the free Lie algebra on the same generating set (Witt [12]). All Lie algebras are over \mathbb{Z} unless otherwise specified, and while they will often have a grading which is respected by the bracket, the bracket does not depend on the grading.

Let $\hat{\pi} := \pi_1(\hat{\Sigma}, *)$, and for each boundary component $P_i \in \partial \hat{\Sigma}$, take $\zeta_i \in \hat{\pi}$ to be a loop passing once around P_i , oriented so that P_i lies on the left side of ζ_i . Define

$$T(\Sigma) := \langle [\hat{\pi}, \hat{\pi}], \zeta_1, \dots, \zeta_k, \zeta_0 \rangle$$

to be the normal subgroup generated by $[\widehat{\pi}, \widehat{\pi}]$ and the boundary loops ζ_i , where $|\widehat{\mathcal{P}}| = |\mathcal{P}| = k + 1$. We often write T for $T(\Sigma)$ when there is no confusion. Consider the central series defined by

$$\Gamma_1^T = \widehat{\pi}, \quad \Gamma_2^T = T, \quad \Gamma_j^T = \langle [\Gamma_1^T, \Gamma_{j-1}^T], [\Gamma_2^T, \Gamma_{j-2}^T] \rangle \text{ for } j \geq 3.$$

This is the minimal filtration for which $\Gamma_2^T = T$ and $[\Gamma_i^T, \Gamma_j^T] \subset \Gamma_{i+j}^T$. Define $\mathcal{L}_j^T := \Gamma_j^T / \Gamma_{j+1}^T$ and $\mathcal{L}^T = \bigoplus \mathcal{L}_j^T$. Note that \mathcal{L}_1^T coincides with $H(\Sigma)$. Similarly, let $N(\Sigma)$ denote \mathcal{L}_2^T . Let $\{\xi_1, \dots, \xi_{2g}, \zeta_1, \dots, \zeta_k\}$ be a basis for $\widehat{\pi}$ such that $\{\xi_i\}$ descends to a basis $\{x_i\}$ for \mathcal{L}_1^T , and let $z_i \in \mathcal{L}_2^T$ be the image of ζ_i . We first claim that \mathcal{L}^T is generated as a Lie algebra by

$$\mathcal{S} := \{x_1, \dots, x_{2g}, z_1, \dots, z_k\}.$$

Indeed, a basis for \mathcal{L}_1^T is given by $\{x_i\}$, and a basis for \mathcal{L}_2^T is given by $\{[x_i, x_j] | i < j\} \cup \{z_i\}$. The generator z_0 may be eliminated by applying a relation of the form

$$[\xi_1, \xi_2] \cdots [\xi_{2g-1}, \xi_{2g}] \zeta_1 \cdots \zeta_k \zeta_0 = 1,$$

which gives $\omega + z_1 + \cdots + z_k + z_0 = 0 \in \mathcal{L}_2^T$. (Here $\omega \in \bigwedge^2 \mathcal{L}_1^T$ represents the symplectic form on $H(\Sigma) = \mathcal{L}_1^T$.) From the definition of Γ_j^T we have that \mathcal{L}_j^T is spanned by $[\mathcal{L}_1^T, \mathcal{L}_{j-1}^T] + [\mathcal{L}_2^T, \mathcal{L}_{j-2}^T]$ for $j \geq 3$; the claim now follows by induction. We will see shortly that in fact \mathcal{L}^T is free with basis \mathcal{S} .

Lemma 3.1. *For any morphism $\iota: \Sigma \rightarrow \Sigma'$, the induced map $\widehat{\iota}: \widehat{\pi} \rightarrow \widehat{\pi}'$ satisfies $\widehat{\iota}(\Gamma_j^T(\Sigma)) \subset \Gamma_j^T(\Sigma')$ for all j .*

Proof. By induction on j , it suffices to show that $\widehat{\iota}(T(\Sigma)) \subset T(\Sigma')$. It is automatic that $\widehat{\iota}([\widehat{\pi}, \widehat{\pi}]) \subset [\widehat{\pi}', \widehat{\pi}']$, so it suffices to check that $\widehat{\iota}(\zeta_i) \in T(\Sigma')$. Van Kampen's theorem gives a basis for $\pi_1(\widehat{\Sigma}', *)$ of the form

$$\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j, \beta_i^j\} \cup \{\zeta_i^l\}$$

for $1 \leq i \leq k$, $1 \leq j \leq n_i$, $1 \leq l \leq m_i$ (here either or both of n_i and m_i may be 0), with the property that

$$\xi_i \mapsto \xi_i \text{ and } \zeta_i \mapsto [\alpha_i^1, \beta_i^1] \cdots [\alpha_i^{n_i}, \beta_i^{n_i}] \zeta_i^1 \cdots \zeta_i^{m_i}.$$

Thus $\widehat{\iota}(\zeta_i) \in T(\Sigma')$, as desired. \square

This shows that any inclusion $\iota: \Sigma \rightarrow \Sigma'$ induces a map $\iota: \mathcal{L}^T(\Sigma) \rightarrow \mathcal{L}^T(\Sigma')$. This map is not always injective; for example, if ι is a simple capping, $\widehat{\iota}(\zeta_1)$ is trivial so $\iota(z_1) = 0$. However, this is essentially the only way that injectivity can fail.

Theorem 3.2. *$\mathcal{L}^T(\Sigma)$ is the free Lie algebra on $\mathcal{S}(\Sigma) = \{x_1, \dots, x_{2g}, z_1, \dots, z_k\}$. Furthermore any morphism $\Sigma \rightarrow \Sigma'$ such that no component of $\widehat{\Sigma}' \setminus \widehat{\Sigma}$ is a disk induces an injection $\mathcal{L}^T(\Sigma) \hookrightarrow \mathcal{L}^T(\Sigma')$.*

Proof. We begin with a special case of the latter claim, which lets us embed \mathcal{L}^T in a larger free Lie algebra. Let S' be the surface obtained from \widehat{S} by attaching a surface of genus 1 to each boundary component of \widehat{S} except the one containing the basepoint, and let $\pi' := \pi_1(S', *)$. Van Kampen's theorem gives an inclusion $\widehat{\iota}: \widehat{\pi} \hookrightarrow \pi'$; we may choose a basis $\{\xi_1, \dots, \xi_{2g}, \alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$ for π' such that $\xi_i \mapsto \xi_i$ and $\zeta_i \mapsto [\alpha_i, \beta_i]$. The map $\iota: \mathcal{L}^T \rightarrow \mathcal{L}'$ sends $x_i \mapsto x_i$ and $z_i \mapsto [a_i, b_i]$, where x_i, a_i, b_i are the classes in \mathcal{L}'_1 of ξ_i, α_i, β_i respectively.

A subset Y of a Lie algebra is called *independent* if the subalgebra generated by Y is free with basis Y . We seek to show $\iota(\mathcal{S}) = \{x_1, \dots, x_{2g}, [a_1, b_1], \dots, [a_k, b_k]\}$ is independent; note that by Shirshov [11] and Witt [13], any subalgebra of the free Lie algebra \mathcal{L}' is itself free (after tensoring with \mathbb{Q}). We need the following elementary lemma.

Lemma 3.3. *Let \mathcal{L} be the free Lie algebra on a set X , $\mathcal{P} = \{P_1, \dots, P_k\}$ a partition of X , and $Y = \{y_1, \dots, y_k\}$ a subset of \mathcal{L} so that each y_i is nonzero and contained in the algebra generated by P_i . Then Y is independent.*

Proof of lemma. Let $\mathcal{L}_{\mathbb{Q}} = \mathcal{L} \otimes \mathbb{Q}$, and identify each y_i with its image in $\mathcal{L}_{\mathbb{Q}}$; since \mathcal{L} is torsion free (Witt [12, Theorem 4]), the map $\mathcal{L} \rightarrow \mathcal{L}_{\mathbb{Q}}$ is an injection. The following theorem is proved by Shirshov in the course of proving [11, Theorem 2]: if for each $y_i \in Y$ we have that the leading term of y_i is not in the subalgebra of $\mathcal{L}_{\mathbb{Q}}$ generated by the leading terms of $Y \setminus \{y_i\}$, then Y is independent as a subset of $\mathcal{L}_{\mathbb{Q}}$.¹ The elimination theorem for free Lie algebras implies that as a vector space, $\mathcal{L}_{\mathbb{Q}}$ splits for each i as the direct sum of the algebra generated by P_i with the ideal generated by $X \setminus P_i$ (see e.g. Bourbaki [1, Chapter II, Section 2.9, Proposition 10]). Since y_i is contained in the former summand, and the algebra generated by $Y \setminus \{y_i\}$ is contained in the latter summand, the condition above is verified. Thus applying Shirshov's result, we conclude that Y is independent in $\mathcal{L}_{\mathbb{Q}}$; since \mathcal{L} is torsion-free, it follows that Y is independent in \mathcal{L} . \square

Since $\iota(\mathcal{S})$ clearly satisfies the conditions of Lemma 3.3 with the partition

$$\mathcal{P} = \{\{x_1\}, \dots, \{x_{2g}\}, \{a_1, b_1\}, \dots, \{a_g, b_g\}\},$$

we conclude that $\iota(\mathcal{L}^T)$ is free with basis $\iota(\mathcal{S})$. Since the generating set \mathcal{S} is taken bijectively to the free basis $\iota(\mathcal{S})$ by ι , it follows from the universal property that \mathcal{S} is a free basis for $\mathcal{L}^T(\Sigma)$; in particular $\mathcal{L}^T(\Sigma)$ is free.

We now consider a general inclusion $\Sigma \rightarrow \Sigma'$ satisfying the condition that no component of $\widehat{\Sigma}' \setminus \widehat{\Sigma}$ is a disk. Recall from Lemma 3.1 that $\pi_1(\widehat{\Sigma}', *)$ has a basis of the form

$$\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j, \beta_i^j\} \cup \{\zeta_i^l\}$$

for $1 \leq i \leq k$, $1 \leq j \leq n_i$, $1 \leq l \leq m_i$, so that

$$\xi_i \mapsto \xi_i \text{ and } \zeta_i \mapsto [\alpha_i^1, \beta_i^1] \cdots [\alpha_i^{n_i}, \beta_i^{n_i}] \zeta_i^1 \cdots \zeta_i^{m_i}.$$

The condition on $\widehat{\Sigma}' \setminus \widehat{\Sigma}$ implies for each i that n_i and m_i are not both zero. We have already proved that $\mathcal{L}^T(\Sigma')$ is free with basis $\{x_i\} \cup \{a_i^j, b_i^j\} \cup \{z_i^l\}$ where i, j, l range as above. The induced map $\mathcal{L}^T(\Sigma) \rightarrow \mathcal{L}^T(\Sigma')$ sends $x_i \mapsto x_i$ and

$$z_i \mapsto [a_i^1, b_i^1] + \cdots + [a_i^{n_i}, b_i^{n_i}] + z_i^1 + \cdots + z_i^{m_i}.$$

As above, the images of the x_i and z_i satisfy the condition of Lemma 3.3. Thus the image of the generating set \mathcal{S} is independent, so by the universal property the map $\mathcal{L}^T(\Sigma) \rightarrow \mathcal{L}^T(\Sigma')$ is injective as desired; this concludes the proof of the theorem. \square

Comparing Γ_j^T . As noted above, any embedding $\Sigma \subset S_0$ into a surface with one boundary component induces a filtration of $\widehat{\pi}$ by restricting the lower central series. The following corollary states that as long as no component of $S_0 \setminus \widehat{S}$ is a disk, the induced filtration is exactly the central series $\Gamma_j^T = \widehat{\pi} \cap \Gamma_j(S_0)$. Indeed we have the following generalization:

Corollary 3.4. *For any embedding $\Sigma \rightarrow \Sigma'$ so that no component of $\widehat{S}' \setminus \widehat{S}$ is a disk, $\Gamma_j^T(\Sigma) = \widehat{\pi} \cap \Gamma_j^T(\Sigma')$.*

¹The leading term of y is the highest degree homogeneous component of y . For an exposition in English of a closely related theorem, see Bryant–Kovács–Stöhr [2].

Proof. Lemma 3.1 states that $\Gamma_j^T(\Sigma) \subset \widehat{i}^{-1}(\Gamma_j^T(\Sigma'))$. By Theorem 3.2 the map ι restricts to an isomorphism $\mathcal{L}^T \cong \iota(\mathcal{L}^T)$, so in particular ι is injective on each \mathcal{L}_j^T . This implies that $\widehat{i}^{-1}(\Gamma_j^T(\Sigma')) \subset \Gamma_j^T(\Sigma)$ for all j . \square

The following corollary of Theorem 3.2 is actually a special case of the theorem: it states that there are no relations among the basis elements $\{x_1, \dots, x_{2g}, z_1, \dots, z_k\}$ in degree 3, while Theorem 3.2 states that there are no relations among them at all. We will use this proposition in Section 5.5 to bound the image of the Johnson homomorphism. Recall that $H(\Sigma) = \mathcal{L}_1^T$ and $N(\Sigma) = \mathcal{L}_2^T$.

Proposition 3.5. *The commutator bracket $\mathcal{L}_1^T \otimes \mathcal{L}_2^T \rightarrow \mathcal{L}_3^T$ induces the short exact sequence*

$$1 \rightarrow \bigwedge^3 H(\Sigma) \rightarrow H(\Sigma) \otimes N(\Sigma) \rightarrow \mathcal{L}_3^T \rightarrow 1.$$

Since $\text{Mod}(\Sigma)$ fixes the boundary components of \widehat{S} , it fixes ζ_i up to conjugacy, and thus acts trivially on z_1, \dots, z_k . By definition, $\mathcal{I}(\Sigma)$ acts trivially on x_1, \dots, x_{2g} . Since $\{x_1, \dots, x_{2g}, z_1, \dots, z_k\}$ generate \mathcal{L}^T , we have the following corollary of Theorem 3.2, which will be used in Section 5.2.

Corollary 3.6. *The action of $\text{Mod}(\Sigma)$ on $\mathcal{L}^T(\Sigma)$ factors through its action on $H(\Sigma)$; in particular, $\mathcal{I}(\Sigma)$ acts trivially on $\mathcal{L}^T(\Sigma)$.*

3.2 $T(S)$ for general subsurfaces

When S is a subsurface of a closed surface S_0 , Corollary 3.4 relates the lower central series on $\pi_1(S_0)$ to the induced filtration on $\pi_1(\widehat{S})$. In this section, we extend this to the induced filtration on $\pi_1(S)$, which is often more useful in applications. However this section is not necessary for the main results of the paper and may be skipped on a first reading.

Let $\pi = \pi_1(S)$ and $\widehat{\pi} = \pi_1(\widehat{S})$. In this section only, let $T(\Sigma)$ be denoted by $T(\widehat{\pi}) < \widehat{\pi}$. Define $T(\pi) < \pi$ as follows. For each subset $P_i \in \mathcal{P}$, let $\zeta'_i \in \pi$ be a loop passing exactly around the boundary components contained in P_i , oriented so that those boundary components lie on the left side of ζ'_i , and so that the subsurface cobounded by ζ'_i and P_i has genus 0. Now define

$$T(\pi) := \langle [\pi, \pi], \zeta'_1, \dots, \zeta'_k, \zeta'_0 \rangle.$$

Considering π as a subgroup of $\widehat{\pi}$, it is clear that $T(\pi) < T(\widehat{\pi})$; we will show that $T(\pi)$ is exactly the intersection $T(\pi) = \pi \cap T(\widehat{\pi})$. We define two central series initially by

$$\Gamma_1^T(\widehat{\pi}) = \widehat{\pi}, \Gamma_2^T(\widehat{\pi}) = T(\widehat{\pi}) \quad \text{and} \quad \Gamma_1^T(\pi) = \pi, \Gamma_2^T(\pi) = T(\pi)$$

respectively, and then in both cases extend by

$$\Gamma_j^T = \langle [\Gamma_1^T, \Gamma_{j-1}^T], [\Gamma_2^T, \Gamma_{j-2}^T] \rangle \text{ for } j \geq 3.$$

By induction, we have $\Gamma_j^T(\pi) < \Gamma_j^T(\widehat{\pi})$ for all j ; we will show that $\Gamma_j^T(\pi)$ is the intersection $\Gamma_j^T(\pi) = \pi \cap \Gamma_j^T(\widehat{\pi})$. As above, let $\mathcal{L}_j^T(\pi)$ and $\mathcal{L}_j^T(\widehat{\pi})$ be the graded quotients of these filtrations, with $\mathcal{L}^T(\pi) = \bigoplus L_j^T(\pi)$ and $\mathcal{L}^T(\widehat{\pi}) = \bigoplus L_j^T(\widehat{\pi})$ the associated graded Lie algebras. The inclusions $\Gamma_j^T(\pi) < \Gamma_j^T(\widehat{\pi})$ descend to a map $\iota: \mathcal{L}^T(\pi) \rightarrow \mathcal{L}^T(\widehat{\pi})$.

Theorem 3.7. *$L^T(\pi)$ is the free Lie algebra on $\mathcal{S}(\pi) = \{x_1, \dots, x_{2g}\} \cup \{a_i^j\}_{i \geq 0}^{j \geq 1} \cup \{y_i\}_{i \geq 1}$ (defined below), and the natural map $\mathcal{L}^T(\pi) \rightarrow \mathcal{L}^T(\widehat{\pi})$ is an injection.*

Proof. Let $m_i = |P_i| - 1$, where $|P_i|$ is the number of components in the subset P_i . Let $\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j\}$ for $0 \leq i \leq k$, $0 \leq j \leq m_i$ be elements of π so that:

- excluding α_0^0 , the set $\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j\}$ is a basis for π ,
- α_i^j is a loop around the j th boundary component in P_i ,
- and we have the equations

$$\zeta'_i = \alpha_i^0 \alpha_i^1 \cdots \alpha_i^{m_i} \quad (1)$$

$$[\xi_1, \xi_2], \dots, [\xi_{2g-1}, \xi_{2g}] \zeta'_1 \cdots \zeta'_k \zeta'_0 = 1. \quad (2)$$

Let x_i and a_i^j be the images of ξ_i and α_i^j in $\mathcal{L}_1^T(\pi)$, and let y_i be the image of ζ'_i in $L_2^T(\pi)$. From (1) and (2) we obtain the relations

$$a_i^0 + a_i^1 \cdots + a_i^{m_i} = 0 \quad \text{in } L_1^T(\pi)$$

$$[x_1, x_2] + \cdots + [x_{2g-1}, x_{2g}] + y_1 + \cdots + y_k + y_0 = 0 \quad \text{in } L_2^T(\pi).$$

We may thus eliminate the generators a_i^0 and y_0 , implying that $\mathcal{L}^T(\pi)$ is generated by $\mathcal{S}(\pi) = \{x_1, \dots, x_{2g}\} \cup \{a_i^j\}_{i \geq 1}^{j \geq 1} \cup \{y_i\}_{i \geq 1}$. Our goal is to prove that $\mathcal{L}^T(\pi)$ is free on this basis.

We can find $\{\beta_i^j\}$ in $\hat{\pi}$ for $0 \leq i \leq k$, $1 \leq j \leq m_i$ so that

$$\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j\} \cup \{\beta_i^j\} \cup \{\zeta_i\} \text{ for } 0 \leq i \leq k, 1 \leq j \leq m_i$$

is a basis for $\hat{\pi}$, and so that

$$\zeta_i = \alpha_i^0 \beta_i^1 \alpha_i^1 \cdots \beta_i^{m_i} \alpha_i^{m_i} \overline{\beta_i^{m_i}} \cdots \overline{\beta_i^1} \quad (3)$$

where $\overline{\beta} = \beta^{-1}$. By Theorem 3.2, $\mathcal{L}^T(\hat{\pi})$ is the free Lie algebra on $\{x_i\} \cup \{a_i^j\} \cup \{b_i^j\}$ in degree 1 and $\{z_i\}$ in degree 2. For any x_i , and for any a_i^j or b_i^j with $i \geq 1$, the map $\mathcal{L}_1^T(\pi) \rightarrow \mathcal{L}_1^T(\hat{\pi})$ sends $x_i \mapsto x_i$, $a_i^j \mapsto a_i^j$, and $b_i^j \mapsto b_i^j$. From the above formulas (1) and (3) for ζ'_i and ζ_i , we see that $\iota(y_i)$ can be expressed in terms of these generators as

$$y_i \mapsto [a_i^1, b_i^1] + [a_i^2, b_i^1 + b_i^2] + \cdots + [a_i^{m_i}, b_i^1 + \cdots + b_i^{m_i}] + z_i.$$

If we can show that the image $\iota(\mathcal{S}(\pi)) = \{x_i\} \cup \{a_i^j\} \cup \{\iota(y_i)\}$ is independent, the universal property will imply that $\mathcal{L}^T(\pi)$ is free on this basis, and that ι is injective. Since $\iota(y_i)$ involves multiple generators, we cannot appeal to Lemma 3.3, but we can apply Shirshov's result directly. Since each element of $\iota(\mathcal{S}(\pi))$ is homogeneous, it suffices to show that for each $w \in \iota(\mathcal{S}(\pi))$, the element w is not contained in the subalgebra of $\mathcal{L}^T(\hat{\pi})$ generated by $\iota(\mathcal{S}(\pi)) \setminus \{w\}$. For the basis elements x_i and a_i^j , this is easy, since no other element of $\iota(\mathcal{S}(\pi))$ involves that basis element. For $\iota(y_i)$, note that no other element of $\iota(\mathcal{S}(\pi))$ involves the basis element z_i , so the subalgebra generated by $\iota(\mathcal{S}(\pi)) \setminus \{\iota(y_i)\}$ cannot contain $\iota(y_i)$. Thus the image of $\mathcal{L}^T(\pi)$ is free with basis $\iota(\mathcal{S}(\pi))$, concluding the proof. \square

4 The Johnson filtration

Let Σ be a partitioned surface, and let $\widehat{\pi} = \pi_1(\widehat{\Sigma}, *)$ as before. We saw before Corollary 3.6 that the action of $\text{Mod}(\Sigma)$ on $\widehat{\pi}$ preserves the central series Γ_i^T defined in Section 3.1. We seek to define the *partitioned Johnson filtration*

$$\text{Mod}(\Sigma) = \text{Mod}_{(1)}(\Sigma) > \text{Mod}_{(2)}(\Sigma) > \text{Mod}_{(3)}(\Sigma) > \dots$$

The classical Johnson filtration for a surface with one boundary component consists of those homeomorphisms acting trivially modulo Γ_k , but for partitioned surfaces we need to impose another condition.

4.1 Action on arcs

For any $P_i \in \mathcal{P}$, let A_i be an arc in $\widehat{\Sigma}$ from the basepoint $*$ to the boundary component z_i corresponding to P_i . For any $\varphi \in \text{Mod}(\Sigma)$, the image $\varphi(A_i)$ is another arc with the same endpoints, so $\varphi(A_i)A_i^{-1}$ may be considered as an element of $\widehat{\pi}$. Let $d_i(\varphi) := \varphi(A_i)A_i^{-1} \in \widehat{\pi}$. Of course this depends on the choice of A_i , but we will see in Lemma 4.2 that its equivalence class under a certain relation is well-defined.

Lemma 4.1. *If φ acts trivially on $\widehat{\pi}/\Gamma_k^T$, then $d_i(\varphi) \in \Gamma_{k-2}^T$.*

Proof. We can write $\zeta_i = A_i Z_i A_i^{-1} \in \widehat{\pi}$, where Z_i is a loop contained in $z_i \subset \partial\widehat{\Sigma}$ (note that Z_i is fixed by φ). We then compute

$$\varphi(\zeta_i)\zeta_i^{-1} = \varphi(A_i)Z_i\varphi(A_i)^{-1}A_iZ_i^{-1}A_i^{-1} = [\varphi(A_i)A_i^{-1}, A_iZ_iA_i^{-1}] = [d_i(\varphi), \zeta_i]. \quad (4)$$

Since φ acts trivially modulo Γ_k^T , we know that $\varphi(\zeta_i)\zeta_i^{-1} \in \Gamma_k^T$. But $\zeta_i \in \Gamma_2^T$ represents the free generator $z_i \in \mathcal{L}_2^T$ by Theorem 3.2, and so $[d_i(\varphi), \zeta_i] \in \Gamma_k^T$ implies $d_i(\varphi) \in \Gamma_{k-2}^T$. \square

Lemma 4.2. *When restricted to the subgroup of $\text{Mod}(\Sigma)$ acting trivially on $\widehat{\pi}/\Gamma_k^T$, this yields a well-defined homomorphism d_i mapping $\varphi \mapsto d_i(\varphi) \in \Gamma_{k-2}^T/\Gamma_{k-1}^T = \mathcal{L}_{k-2}^T$.*

Proof. We need to verify that the class of $d_i(\varphi)$ in \mathcal{L}_{k-2}^T does not depend on the arc $A = A_i$ chosen. Assume that φ acts trivially on $\widehat{\pi}/\Gamma_k^T$, and let B be another such arc. Then

$$(\varphi(A)A^{-1})(\varphi(B)B^{-1})^{-1} = \varphi(AB^{-1})(AB^{-1})^{-1}[AB^{-1}, \varphi(B)B^{-1}] = \varphi(x)x^{-1}[x, \varphi(B)B^{-1}]$$

where $x = AB^{-1} \in \widehat{\pi}$. Since φ acts trivially modulo Γ_k^T , the first term $\varphi(x)x^{-1}$ lies in Γ_k^T . By Lemma 4.1, $\varphi(B)B^{-1} \in \Gamma_{k-2}^T$, so $[x, \varphi(B)B^{-1}] \in \Gamma_{k-1}^T$. Thus $\varphi(A)A^{-1}$ and $\varphi(B)B^{-1}$ agree modulo Γ_{k-1}^T , as desired. \square

Lemma 4.3. *If $|P_i| > 1$, then for any φ acting trivially on $\widehat{\pi}/\Gamma_k^T$ we have $d_i(\varphi) = 0 \in \mathcal{L}_{k-2}^T$.*

Proof. Let $\alpha = \alpha_i^j$ be a loop around a boundary component contained in P_i (the notation is as in Theorem 3.7). We can write $\alpha = A_i \nu A_i^{-1}$, where ν is contained in ∂S and thus is fixed by φ . Then just as in (4) we have $\varphi(\alpha)\alpha^{-1} = [d_i(\varphi), \alpha]$, and by assumption this is contained in Γ_k^T . However in Theorem 3.7 we saw that $\alpha \in \widehat{\pi}$ represents the free generator $\alpha_i^j \in \mathcal{L}_1^T$, and so $[d_i(\varphi), \alpha] \in \Gamma_k^T$ implies that $d_i(\varphi) \in \Gamma_{k-1}^T$. Thus $d_i(\varphi)$ vanishes in $\mathcal{L}_{k-2}^T = \Gamma_{k-2}^T/\Gamma_{k-1}^T$. \square

4.2 The partitioned Johnson filtration

Definition 4.4 (The partitioned Johnson filtration). Let $\text{Mod}_{(k)}(\Sigma)$ be the subgroup of $\text{Mod}(\Sigma)$ consisting of those $\varphi \in \text{Mod}(\Sigma)$ satisfying

- (I) φ acts trivially on $\widehat{\pi}$ modulo Γ_k^T , and
- (II) for each $P_i \in \mathcal{P}$ with $|P_i| = 1$, the element $\varphi(A_i)A_i^{-1}$ is contained in Γ_{k-1}^T .

By Lemma 4.2 and Lemma 4.3, the second condition is equivalent to the condition that $d_i(\varphi) = 0 \in \mathcal{L}_{k-2}^T$ for all i .

The first terms of the Johnson filtration. Note that $\text{Mod}_{(1)}(\Sigma) = \text{Mod}(\Sigma)$ by definition, and $\text{Mod}_{(2)}(\Sigma)$ is the Torelli group $\mathcal{I}(\Sigma)$ defined in Section 2.2. We denote the next term $\text{Mod}_{(3)}(\Sigma)$ by $\mathcal{K}(\Sigma)$ and call it the *Johnson kernel of Σ* . In Section 5 we will define the *partitioned Johnson homomorphism* τ_Σ , and prove in Theorem 5.5 that $\mathcal{K}(\Sigma) = \ker \tau_\Sigma$. In particular, this will show that $\mathcal{K}(\Sigma)$ is exactly the subgroup of $\text{Mod}(\Sigma)$ acting trivially on $\widehat{\pi}$ modulo $\Gamma_3^T = [\widehat{\pi}, T]$; condition (II) in Definition 4.4 is not necessary in this case.

Changing the basepoint.

Theorem 4.5. *The Johnson filtration does not depend on the basepoint; that is, if Σ and Σ' differ only in the location of the basepoint, then $\text{Mod}_{(k)}(\Sigma) = \text{Mod}_{(k)}(\Sigma')$.*

Proof. If $\Sigma = (S, \mathcal{P}, *)$, let $\Sigma' = (S, \mathcal{P}, *')$. An isomorphism from $\widehat{\pi} = \pi_1(\widehat{\Sigma}, *)$ to $\widehat{\pi}' = \pi_1(\widehat{\Sigma}, *')$ is given by $x \mapsto A^{-1}xA$, where $A = A_i$ is an arc from $*$ to $*' \in P_i$ as before. This isomorphism takes the central series $\Gamma_k^T(\Sigma)$ of $\widehat{\pi}$ to the central series $\Gamma_k^T(\Sigma')$ of $\widehat{\pi}'$ (to check this, it suffices to check that $A^{-1}\zeta_i A \in T(\widehat{\pi}')$). In this proof only, let \bar{x} denote $A^{-1}xA$. We compute

$$\begin{aligned} \varphi(\bar{x})\bar{x}^{-1} &= \varphi(A^{-1}xA)A^{-1}x^{-1}A \\ &= [\varphi(A^{-1})A, A^{-1}\varphi(x)A]A^{-1}\varphi(x)x^{-1}A \\ &= [d_i(\varphi)^{-1}, \varphi(x)]\varphi(x)x^{-1} = \overline{[d_i(\varphi)^{-1}, \varphi(x)]\varphi(x)x^{-1}} \end{aligned} \tag{5}$$

If $\varphi \in \text{Mod}_{(k)}(\Sigma)$ we know that $\varphi(x)x^{-1} \in \Gamma_k^T(\Sigma)$, and furthermore $d_i(\varphi) \in \Gamma_{k-1}^T(\Sigma)$, so $[d_i(\varphi)^{-1}, \varphi(x)] \in \Gamma_k^T(\Sigma)$ as well. This shows that φ acts trivially on $\widehat{\pi}'$ modulo $\Gamma_k^T(\Sigma')$, verifying condition (I) of Definition 4.4. For condition (II), note that we may take the arc A'_j from $*'$ to the boundary component corresponding to P_j to be $A'_j = A_i^{-1}A_j$. We compute:

$$\varphi(A'_j)A_j'^{-1} = \varphi(A_i)^{-1}\varphi(A_j)A_j^{-1}A_i = \overline{A_i\varphi(A_i)^{-1}\varphi(A_j)A_j^{-1}} = \overline{d_i(\varphi)^{-1}d_j(\varphi)}$$

Since $d_i(\varphi)$ and $d_j(\varphi)$ lie in $\Gamma_{k-1}^T(\Sigma)$, we conclude that $\varphi(A'_j)A_j'^{-1}$ lies in $\Gamma_{k-1}^T(\Sigma')$, verifying condition (II). \square

4.3 The Johnson filtration is preserved by inclusions

For any inclusion $\Sigma \rightarrow \Sigma'$ such that no component of $\widehat{\Sigma}' \setminus S$ is a disk or annulus with boundary contained in ∂S , the induced map $\text{Mod}(\Sigma) \rightarrow \text{Mod}(\Sigma')$ is an injection (see e.g. Paris–Rolfen [9, Corollary 4.2(iii)]).

Theorem 4.6. *Let $\Sigma \rightarrow \Sigma'$ be any inclusion such that no component of $\widehat{\Sigma}' \setminus S$ is a disk or annulus with boundary contained in ∂S . Then for any $k \geq 1$, we have $\text{Mod}_{(k)}(\Sigma) = \text{Mod}(\Sigma) \cap \text{Mod}_{(k)}(\Sigma')$.*

Proof. As in Lemma 3.1, choose a basis for $\widehat{\pi}'$ of the form

$$\{\xi_1, \dots, \xi_{2g}\} \cup \{\alpha_i^j, \beta_i^j\} \cup \{\zeta_i^l\}$$

so that

$$\{\xi_i\} \cup \{\zeta_i = [\alpha_i^1, \beta_i^1] \cdots [\alpha_i^{n_i}, \beta_i^{n_i}] \zeta_i^1 \cdots \zeta_i^{m_i}\}$$

form a basis for $\widehat{\pi}$.

We first assume that $\varphi \in \text{Mod}_{(k)}(\Sigma)$ and seek to show that $\varphi \in \text{Mod}_{(k)}(\Sigma')$. Since $\{\xi_i\} \cup \{\alpha_i^j, \beta_i^j\} \cup \{\zeta_i^l\}$ generate the central series $\Gamma_j^T(\Sigma')$, in showing that φ lie in $\text{Mod}_{(k)}(\Sigma')$, for condition (I) it suffices to show that $\varphi(\eta)\eta^{-1} \in \Gamma_k^T(\Sigma)$ for any η in this basis. For ξ_i this is automatic, since $\xi_i \in \widehat{\pi}$: the assumption that $\varphi \in \text{Mod}_{(k)}(\Sigma)$ implies by condition (I) that $\varphi(\xi_i)\xi_i^{-1} \in \Gamma_k^T$, and by Lemma 3.1 $\widehat{i}(\Gamma_k^T(\Sigma)) \subset \Gamma_k^T(\Sigma')$. For $\eta \in \{\alpha_i^j, \beta_i^j\} \cup \{\zeta_i^l\}$, however, we can write $\eta_i = A_i H A_i^{-1} \in \widehat{\pi}'$, where H is a loop contained entirely in $\widehat{\Sigma}' \setminus \widehat{\Sigma}$. In particular H is fixed by φ , so we can compute just as in (4) that $\varphi(\eta)\eta^{-1} = [d_i(\varphi), \eta]$. But the assumption that $\varphi \in \text{Mod}_{(k)}(\Sigma)$ implies by condition (II) that $d_i(\varphi) = \varphi(A_i)A_i^{-1} \in \Gamma_{k-1}^T(\Sigma)$. Applying Lemma 3.1 we conclude that $\varphi(\eta)\eta^{-1} = [d_i(\varphi), \eta]$ is contained in $\Gamma_k^T(\Sigma')$, as desired. To check that φ lies in $\text{Mod}_{(k)}(\Sigma')$ it remains to check condition (II). But for any boundary component P'_j of $\widehat{\Sigma}'$, we may choose the arc A'_j from the basepoint to P'_j so that it meets $\widehat{\Sigma}$ in the arc A_i for some i . Then $\varphi(A'_j)A'_j$ essentially coincides with $\varphi(A_i)A_i^{-1}$, which we saw above is contained in $\Gamma_{k-1}^T(\Sigma)$; the only difference is possibly in the location of the basepoint, and we conclude that $d'_j(\varphi) \in \Gamma_{j-1}^T(\Sigma')$, as desired.

We now assume that $\varphi \in \text{Mod}(\Sigma) \cap \text{Mod}_{(k)}(\Sigma')$ and seek to show that $\varphi \in \text{Mod}_{(k)}(\Sigma)$. Condition (I) is easy to check. The conditions on $\Sigma \rightarrow \Sigma'$ implies that $\widehat{\pi} \rightarrow \widehat{\pi}'$ is injective. For any $x \in \widehat{\pi}$, the assumption that $\varphi \in \text{Mod}_{(k)}(\Sigma')$ implies that $\varphi(x)x^{-1} \in \Gamma_k^T(\Sigma')$. But certainly $\varphi(x)x^{-1} \in \widehat{\pi}$, and Corollary 3.4 implies that $\widehat{\pi} \cap \Gamma_k^T(\Sigma') = \Gamma_k^T(\Sigma)$. For condition (II), fix a component P_i of $\widehat{\Sigma}$; we seek to show that $d_i(\varphi) = 0$. Let T be the component of $\widehat{\Sigma}' \setminus \widehat{\Sigma}$ meeting $\widehat{\Sigma}$ in P_i . If T has another boundary component P'_j not contained in $\widehat{\Sigma}$, we saw above that $d_i(\varphi) = d'_j(\varphi)$; in particular, the assumption that $\varphi \in \text{Mod}_{(k)}(\Sigma')$ implies that $d'_j(\varphi) = d_i(\varphi)$ vanishes in $\mathcal{L}_{k-2}^T(\Sigma)$. It remains to consider the case when $\partial T = P_i$. But then the assumption on $\widehat{\Sigma}' \setminus S$ implies that $|P_i| > 2$, so by Lemma 4.3 we have $d_i(\varphi) = 0$ as desired. \square

5 The Johnson homomorphism

In this section and the next, we construct a surjective homomorphism $\tau_\Sigma: \mathcal{I}(\Sigma) \rightarrow W_\Sigma$ whose kernel is exactly $\mathcal{K}(\Sigma)$.

5.1 The classical Johnson homomorphism

We briefly review Johnson's original construction of the Johnson homomorphism in the case when S has one boundary component, using the action of $\text{Mod}(S)$ on the universal two-step nilpotent quotient of the free group $\pi := \pi_1(S, *)$, where $*$ $\in \partial S$. Let $[\pi, \pi]$ be the commutator subgroup of π ; we have the short exact sequence

$$1 \rightarrow [\pi, \pi] \rightarrow \pi \rightarrow H \rightarrow 1. \quad (6)$$

Centralizing the first term, we get the short exact sequence

$$1 \rightarrow N \rightarrow E \rightarrow H \rightarrow 1$$

of Johnson [7], where $H = \pi/\pi_2$, $N = \pi_2/\pi_3$, and $E = \pi/\pi_3$.

Considering the exact sequence (6) as a presentation for H , Hopf's formula says that

$$H_2(H) = \frac{\pi_2 \cap [\pi, \pi]}{[\pi_2, \pi]} = \frac{\pi_2}{\pi_3} = N.$$

Identifying $H_2(H)$ with $\bigwedge^2 H$, we have an isomorphism $\bigwedge^2 H \cong N$, which can be given explicitly as follows: $x \wedge y \in \bigwedge^2 H$ is sent to $[\tilde{x}, \tilde{y}] \in N$, where $\tilde{x}, \tilde{y} \in E$ are lifts of x and y . All these identifications are $\text{Mod}(S)$ -equivariant; in particular, $\mathcal{I}(S)$ acts trivially on N .

Definition 5.1. The Johnson homomorphism $\tau: \mathcal{I}(\Sigma) \rightarrow \text{Hom}(H, N)$ is defined by assigning to $\varphi \in \mathcal{I}(\Sigma)$ the homomorphism $\tau(\varphi): H \rightarrow N$ given by $x \mapsto [f(\tilde{x})\tilde{x}^{-1}]$, where $x \in \pi$ is any lift of $x \in H$. The fact that N is central in E implies that $\tau(\varphi)$ is a homomorphism, and the fact that $\mathcal{I}(\Sigma)$ acts trivially on H and on N implies that τ is a homomorphism.

Identifying N with $\bigwedge^2 H$ and $\text{Hom}(H, N)$ with $H^* \otimes \bigwedge^2 H$, Johnson proved in [4] that the image of this map is exactly $\bigwedge^3 H \leq H^* \otimes \bigwedge^2 H$; compare with Theorem 5.7. The map τ is $\text{Mod}(S)$ -equivariant with respect to the conjugation actions on $\mathcal{I}(S)$ and on $\text{Hom}(H, \bigwedge^2 H)$. The kernel $\ker \tau \leq \mathcal{I}(S)$ is exactly the subgroup acting trivially on $E = \pi/\pi_3$. Johnson proved in [7] that for a surface with at most one boundary component, $\ker \tau$ is the group generated by separating twists.

5.2 The partitioned Johnson homomorphism

Our construction of the Johnson homomorphism for a general partitioned surface follows this approach closely. We remark that for the remainder of Section 5, we only deal with $\widehat{\Sigma}$, not Σ itself; as a result, the reader may assume that the partition on Σ is totally separated, so that $\Sigma = \widehat{\Sigma}$, without contradiction.

As in Section 3.1, let $\widehat{\pi} := \pi_1(\widehat{\Sigma}, *)$, and for each boundary component $P_i \in \partial\widehat{\Sigma}$, take $\zeta_i \in \widehat{\pi}$ to be a loop passing once around P_i , oriented so that P_i lies on the left side of ζ_i . As before

$$T(\Sigma) := \langle [\widehat{\pi}, \widehat{\pi}], \zeta_1, \dots, \zeta_k, \zeta_0 \rangle$$

is the normal subgroup generated by $[\widehat{\pi}, \widehat{\pi}]$ and the boundary loops ζ_i , where $|\widehat{\mathcal{P}}| = |\mathcal{P}| = k + 1$. The definition of $T(\Sigma)$ is motivated by Theorem 3.2 and Corollary 3.4; in particular, if $\Sigma \subset S_0$ is an inclusion into a surface with one boundary component and $\pi_0 = \pi_1(S_0)$, we have $T(\Sigma) = \widehat{\pi} \cap [\pi_0, \pi_0]$. We often write T for $T(\Sigma)$ when there is no confusion. Note that the quotient of $\widehat{\pi}$ by T gives an exact sequence

$$1 \rightarrow T \rightarrow \widehat{\pi} \rightarrow H(\Sigma) \rightarrow 1. \quad (7)$$

Centralizing the first term, we get the exact sequence

$$1 \rightarrow T/[T, \widehat{\pi}] \rightarrow \widehat{\pi}/[T, \widehat{\pi}] \rightarrow H(\Sigma) \rightarrow 1,$$

which we label as

$$1 \rightarrow N(\Sigma) \rightarrow E(\Sigma) \rightarrow H(\Sigma) \rightarrow 1.$$

Definition 5.2 (The partitioned Johnson homomorphism). The Torelli group $\mathcal{I}(\Sigma)$ acts trivially on $N(\Sigma)$ by Corollary 3.6, and on $H(\Sigma)$ by definition. Thus by the construction described in Definition 5.1, the action of $\mathcal{I}(\Sigma)$ on $E(\Sigma) = \widehat{\pi}/[T, \widehat{\pi}]$ yields a homomorphism

$$\tau_\Sigma: \mathcal{I}(\Sigma) \rightarrow \text{Hom}(H(\Sigma), N(\Sigma))$$

which we call the (*partitioned*) *Johnson homomorphism*. It is given explicitly by

$$\tau_\Sigma(\varphi)(x) = [\varphi(\tilde{x})\tilde{x}^{-1}]$$

where $\tilde{x} \in \hat{\pi}$ is a lift of $x \in H(\Sigma)$.

Note that $\tau_\Sigma(\varphi) = 0$ if and only if φ acts trivially on $E(\Sigma) = \hat{\pi}/\Gamma_3^T$. The map τ_Σ is $\text{Mod}(\Sigma)$ -equivariant, in the following sense. The mapping class group $\text{Mod}(\Sigma)$ acts on $\mathcal{I}(\Sigma)$ by conjugation. The action of $\text{Mod}(\Sigma)$ on $H(\Sigma)$ and $N(\Sigma)$ induces an action on $\text{Hom}(H(\Sigma), N(\Sigma))$; to be precise, if $f \in \text{Hom}(H(\Sigma), N(\Sigma))$ and $\varphi \in \text{Mod}(\Sigma)$, the map φ_*f is defined by $\varphi_*f(x) = \varphi(f(\varphi^{-1}(x)))$. The following is a formal consequence of the definition of τ_Σ .

Lemma 5.3. *Let $\varphi \in \text{Mod}(\Sigma)$ and $\psi \in \mathcal{I}(\Sigma)$. Then $\tau_\Sigma(\varphi\psi\varphi^{-1}) = \varphi_*\tau_\Sigma(\psi)$.*

Proof. From the definitions, we have

$$\begin{aligned} \tau_\Sigma(\varphi\psi\varphi^{-1})(x) &= [\varphi\psi\varphi^{-1}(\tilde{x}) \cdot \tilde{x}^{-1}] \\ &= \varphi[\psi\varphi^{-1}(\tilde{x}) \cdot \varphi^{-1}(\tilde{x})^{-1}] \\ &= \varphi(\tau_\Sigma(\psi)(\varphi^{-1}(x))) \\ &= \varphi_*\tau_\Sigma(\psi)(x) \end{aligned}$$

for any $x \in H(\Sigma)$. □

5.3 Action on arcs

In Lemma 4.2 we used the action of $\mathcal{I}(\Sigma)$ on arcs connecting different boundary components to construct a family of abelian quotients d_i of $\mathcal{I}(\Sigma)$. Recall that $d_i: \mathcal{I}(\Sigma) \rightarrow H(\Sigma)$ is defined by

$$d_i(\varphi) = [\varphi(A_i)A_i^{-1}] \in H(\Sigma),$$

where A_i is an arc from the basepoint $*$ to the boundary component z_i corresponding to P_i . One important difference between $\mathcal{I}(\Sigma)$ and other terms in the Johnson filtration is that these maps d_i factor through the Johnson homomorphism τ_Σ ; this is proved in Section 5.5.

For each P_i we obtain a map $N(\Sigma) \rightarrow \mathbb{Z}$ by intersecting elements with the arcs A_i . Specifically, if $y \in N(\Sigma)$ is represented by $\tilde{y} \in \hat{\pi}$, let (y, A_i) be the algebraic intersection number of \tilde{y} with the arc A_i . This does not depend on the choice of arc A_i . Indeed, two such arcs for the same i differ by a cycle, which has trivial intersection with any commutator or any boundary component ζ_j , and $N(\Sigma)$ is generated by $[\hat{\pi}, \hat{\pi}]$ and the ζ_j .

Definition 5.4. The homomorphism

$$\delta_i: \text{Hom}(H(\Sigma), N(\Sigma)) \rightarrow H(\Sigma)$$

is defined by the adjunction

$$(f(x), A_i) = (x, \delta_i(f)). \tag{8}$$

The following proposition will be proved in Section 5.5.

Theorem 5.5. *The maps $d_i: \mathcal{I}(\Sigma) \rightarrow H(\Sigma)$ factor through τ_Σ ; more specifically, we have $d_i = \delta_i \circ \tau_\Sigma$.*

5.4 The image of τ_Σ

Let $|\mathcal{P}| = k + 1$. There is a natural quotient $N(\Sigma) \twoheadrightarrow \bigwedge^2 H(\Sigma)$ defined by $z_i \mapsto 0$ for $1 \leq i \leq k$, induced for example by the inclusion $\widehat{\Sigma} \hookrightarrow \overline{\Sigma}$ of Section 2.2. In fact, it follows from Theorem 3.2 that

$$N(\Sigma) \simeq \bigwedge^2 H(\Sigma) \oplus \mathbb{Z}^k. \quad (9)$$

where $\bigwedge^2 H(\Sigma)$ is the image of $[\widehat{\pi}, \widehat{\pi}]$ and the \mathbb{Z}^k factor is spanned by z_1, \dots, z_k . Note that the intersection $y \mapsto (y, A_i)$ vanishes on $\bigwedge^2 H(\Sigma)$ and satisfies $(z_j, A_i) = \delta_{ij}$.

The projection $N(\Sigma) \twoheadrightarrow \bigwedge^2 H(\Sigma)$ induces a projection:

$$\mathrm{Hom}(H(\Sigma), N(\Sigma)) \twoheadrightarrow \mathrm{Hom}(H(\Sigma), \bigwedge^2 H(\Sigma)) \simeq H(\Sigma) \otimes \bigwedge^2 H(\Sigma) \quad (10)$$

Note that $\bigwedge^3 H(\Sigma)$ embeds into $H(\Sigma) \otimes \bigwedge^2 H(\Sigma)$ as the ‘‘Jacobi identity’’:

$$a \wedge b \wedge c \mapsto a \otimes b \wedge c + b \otimes a \wedge c + c \otimes a \wedge b$$

Finally, let $D(\Sigma) \leq H(\Sigma)$ be the isotropic subspace spanned by the homology classes of the boundary components; similarly, let $D_i \leq D(\Sigma)$ be the subspace spanned by those components in $P_i \in \mathcal{P}$. Note that $D(\Sigma)^\perp$ is exactly the subspace of $H(\Sigma)$ spanned by $H_1(S)$.

Definition 5.6. The subspace $W_\Sigma \leq \mathrm{Hom}(H(\Sigma), N(\Sigma))$ consists of those elements $f: H(\Sigma) \rightarrow N(\Sigma)$ satisfying the following conditions:

- (I) the image in $H(\Sigma) \otimes \bigwedge^2 H(\Sigma)$ of f under the projection (10) is contained in the subspace $\bigwedge^3 H(\Sigma) \leq H(\Sigma) \otimes \bigwedge^2 H(\Sigma)$.
- (II) for $i \geq 1$, $\delta_i(f) \in D(\Sigma)^\perp$ and furthermore for any $a \in D_i$, $f(a) = \delta_i(f) \wedge a$.
- (III) for any $a \in D_0$, $f(a) = 0$.

The following characterization of the image of τ_Σ is one of the main theorems of the paper.

Theorem 5.7. $W_\Sigma = \mathrm{im} \tau_\Sigma$.

We will prove that $\mathrm{im} \tau_\Sigma$ is contained in W_Σ in Theorem 5.9. We will defer the proof of Theorem 5.7 until Section 6, where we first establish formulas for the value of τ_Σ on various fundamental elements of $\mathcal{I}(\Sigma)$, then use these computations to prove that the image of τ_Σ is all of W_Σ .

Remark 5.8. W_Σ is abstractly isomorphic to $\bigwedge^3 D(\Sigma)^\perp \oplus (D(\Sigma)^\perp)^k$ as a $\mathrm{Mod}(\Sigma)$ -module. To see this, first consider the projection $\mathrm{Hom}(H(\Sigma), N(\Sigma)) \twoheadrightarrow H(\Sigma)^k$ defined by $f \mapsto (\delta_1(f), \dots, \delta_k(f))$. When restricted to W_Σ , this projection has image $(D(\Sigma)^\perp)^k$ by condition (II).

The kernel of this surjection $W_\Sigma \twoheadrightarrow (D(\Sigma)^\perp)^k$ is those elements with $\delta_i(f) = 0$ for all i . By condition (I) such a map lies in $\bigwedge^3 H(\Sigma) < \mathrm{Hom}(H(\Sigma), \bigwedge^2 H(\Sigma))$, and by conditions (II) and (III) satisfies $f(a) = 0$ whenever $a \in D(\Sigma)$. As an element of $\bigwedge^3 H(\Sigma)$, this means that f is contained in

$$(D(\Sigma)^\perp \otimes \bigwedge^2 H(\Sigma)) \cap \bigwedge^3 H(\Sigma) \simeq \bigwedge^3 D(\Sigma)^\perp.$$

Note however that the decomposition $W_\Sigma \simeq \bigwedge^3 D(\Sigma)^\perp \oplus (D(\Sigma)^\perp)^k$ is not canonical; the subspace $\bigwedge^3 D(\Sigma)^\perp < W_\Sigma$ is well-defined, but its complement is not.

5.5 Restricting the image of τ_Σ

Theorem 5.9. *The map τ_Σ has image contained in W_Σ .*

Proof of Theorem 5.5 and Theorem 5.9. Consider $\varphi \in \mathcal{I}(\Sigma)$ and let $f = \tau_\Sigma(\varphi) \in \text{Hom}(H(\Sigma), N(\Sigma))$. We first show that f satisfies condition (I), and at the same time we will verify Theorem 5.5.

We will make use of the following identities (where ${}^a b$ denotes conjugation):

$$[aw, b] = {}^a[w, b] \cdot [a, b], \quad [a, bw] = [a, b] \cdot {}^b[a, w].$$

If $w \in \Gamma_2^T = T$, both $[w, b]$ and $[a, w]$ are in Γ_3^T , so we have

$$[aw, b] \equiv [a, b][w, b] \pmod{\Gamma_4^T}, \quad [a, bw] \equiv [a, b][a, w] \pmod{\Gamma_4^T}.$$

The key to our proof is to consider the action of φ on ζ_0 — even though ζ_0 is contained in the boundary $\partial\widehat{S}$ and so the action of φ on ζ_0 is trivial. Choose a basis so that $\zeta_0 = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \zeta_1 \cdots \zeta_k$. Let a_i and b_i be the homology classes of α_i and β_i respectively; note that $\{a_i, b_i\}$ form a symplectic basis of $H(\Sigma)$.

We are now ready to consider $\varphi(\zeta_0)$. Let $\eta_\varphi(x) := x^{-1}\varphi(x)$. Note that $\varphi \in \mathcal{I}(\Sigma)$ implies that $\eta_\varphi(x) \in \Gamma_2^T$ for all $x \in \widehat{\pi}$ (in fact, these are equivalent), and $\zeta_j \in \Gamma_2^T$ for all j . Recall from (4) that $\varphi(\zeta_j) = [d_j(\varphi), \zeta_j]\zeta_j$. We calculate:

$$\begin{aligned} \varphi(\zeta_0) &= \varphi \left(\prod_{i=1}^g [\alpha_i, \beta_i] \prod_{j=1}^k \zeta_j \right) \\ &= \prod_{i=1}^g [\varphi(\alpha_i), \varphi(\beta_i)] \prod_{j=1}^k \varphi(\zeta_j) \\ &= \prod_{i=1}^g [\alpha_i \eta_\varphi(\alpha_i), \beta_i \eta_\varphi(\beta_i)] \prod_{j=1}^k [d_j, \zeta_j] \zeta_j \\ &\equiv \prod_{i=1}^g [\alpha_i, \beta_i] [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i] \prod_{j=1}^k [d_j, \zeta_j] \zeta_j \pmod{\Gamma_4^T} \\ &\equiv \left(\prod_{i=1}^g [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i] \prod_{j=1}^k [d_j, \zeta_j] \right) \cdot \zeta_0 \pmod{\Gamma_4^T} \end{aligned}$$

Define

$$X = \prod_{i=1}^g [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i] \prod_{j=1}^k [d_j, \zeta_j] \in \Gamma_3^T$$

and consider the class $[X] \in \mathcal{L}_3^T = \Gamma_3^T / \Gamma_4^T$. Note that $\eta_\varphi(\alpha_i)$ and $\eta_\varphi(\beta_i)$ in Γ_2^T represent $\tau(\varphi)(a_i)$ and $\tau(\varphi)(b_i)$ in $N(\Sigma)$, and similarly $\zeta_j \in \Gamma_2^T$ represents $z_j \in N(\Sigma)$. Thus the following element $Y \in H(\Sigma) \otimes N(\Sigma)$ descends to $[X] \in \mathcal{L}_3^T$ under the commutator bracket:

$$Y = \sum_{i=1}^g a_i \otimes \tau_\Sigma(\varphi)(b_i) - b_i \otimes \tau_\Sigma(\varphi)(a_i) + \sum_{j=1}^k d_j(\varphi) \otimes z_j \in H(\Sigma) \otimes N(\Sigma) \quad (11)$$

Note that the first summation is exactly the expansion of $-\tau_\Sigma(\varphi) \in \text{Hom}(H(\Sigma), N(\Sigma))$ in $H(\Sigma) \otimes N(\Sigma)$. Indeed, since $\{a_i, b_i\}$ form a basis of $H(\Sigma)$, we can write

$$\tau_\Sigma(\varphi) = \sum_{i=1}^g a_i^* \otimes \tau_\Sigma(\varphi)(a_i) + b_i^* \otimes \tau_\Sigma(\varphi)(b_i) \in H(\Sigma)^* \otimes N(\Sigma)$$

and under the isomorphism $H(\Sigma) \simeq H(\Sigma)^*$ we have $a_i^* = b_i$ and $b_i^* = -a_i$ (since $a_i^* = (\cdot, b_i)$ and $b_i^* = (\cdot, -a_i)$). In particular, it follows from the discussion following (9) that the coefficient of z_j in the first summation is $-\delta_j(\tau_\Sigma(\varphi)) \otimes z_j$.

We calculated above that $\varphi(\zeta_0) \equiv X \cdot \zeta_0 \pmod{\Gamma_4^T}$. However, since ζ_0 is contained in the boundary of \hat{S} , we have $\varphi(\zeta_0) = \zeta_0$. Thus X must vanish modulo Γ_4^T , and so $[X] = 0 \in \mathcal{L}_3^T$. Thus $Y \in H(\Sigma) \otimes N(\Sigma)$ is contained in the kernel of the map $H(\Sigma) \otimes N(\Sigma) \rightarrow \mathcal{L}_3^T$. We now recall Proposition 3.5, which states that the bracket $\mathcal{L}_1^T \otimes \mathcal{L}_2^T \rightarrow \mathcal{L}_3^T$ induces the short exact sequence

$$1 \rightarrow \bigwedge^3 H(\Sigma) \rightarrow H(\Sigma) \otimes N(\Sigma) \rightarrow \mathcal{L}_3^T \rightarrow 1.$$

This has the following implications.

First, we saw above that the coefficient of z_j in Y is

$$(d_i(\varphi) - \delta_i(\tau_\Sigma(\varphi))) \otimes z_j.$$

But the factor $H(\Sigma) \otimes \mathbb{Z}^k < H(\Sigma) \otimes N(\Sigma)$ intersects $\bigwedge^3 H(\Sigma)$ trivially, so for Y to be contained in $\bigwedge^3 H(\Sigma)$ we must have $d_i(\varphi) - \delta_i(\tau_\Sigma(\varphi)) = 0$. This completes the proof of Theorem 5.5.

Furthermore, this implies that $Y \in H(\Sigma) \otimes N(\Sigma)$ is the difference of $-\tau_\Sigma(\varphi)$ and its projection to the \mathbb{Z}^k factor spanned by the z_j ; in other words, $Y \in H(\Sigma) \otimes \bigwedge^2 H(\Sigma)$ is the projection of $-\tau_\Sigma(\varphi)$ to $\text{Hom}(H(\Sigma), \bigwedge^2 H(\Sigma))$ under the map (10). Now Proposition 3.5 states that this is contained in $\bigwedge^3 H(\Sigma)$, verifying condition (I).

Now we can use Theorem 5.5 to verify conditions (II) and (III). We have just proved that $\delta_i(f) = d_i(\varphi)$. But from the definition of d_i we can see that $d_i(\varphi)$ is contained in $D(\Sigma)^\perp$. Indeed, we have $d_i(\varphi) = [\varphi(A_i)A_i^{-1}]$. Since φ is the identity outside S , we have $\varphi(A_i) = A_i$ outside S , and thus $[\varphi(A_i)A_i^{-1}]$ may be represented by a cycle lying inside S . We observed in Section 5.4 that the span of $H_1(S)$ in $H(\Sigma)$ is exactly $D(\Sigma)^\perp$, and so we conclude that $\delta_i(f) = d_i(\varphi) \in D(\Sigma)^\perp$ for all i .

To verify the remainder of condition (II) we must show that $\tau_\Sigma(\varphi)(a) = d_i(\varphi) \wedge a$ for $a \in D_i$. It suffices to check this when a is the class of a single boundary component in P_i . We saw in Lemma 4.3 that $\varphi(\alpha)\alpha^{-1} = [d_i(\varphi), \alpha]$, where $\alpha \in \hat{\pi}$ represents $a \in D_i$, so $\tau_\Sigma(\varphi)(a) = d_i(\varphi) \wedge a$ as desired. A similar argument verifies condition (III). If a is the class of a single boundary component contained in P_0 , then it is connected outside S to the basepoint; in particular, a can be represented by a loop disjoint from \hat{S} . Thus φ fixes this loop pointwise, and so $\tau_\Sigma(\varphi)(a) = 0$. \square

Remark 5.10. A similar argument can be applied to any term in the Johnson filtration to show that the homomorphisms $d_i: \text{Mod}_{(k)}(\Sigma) \rightarrow \mathcal{L}_{k-1}^T$ are controlled to some degree by the action of $\text{Mod}_{(k)}(\Sigma)$ on $\hat{\pi}/\Gamma_{k+1}^T$. However, to show that the d_i factor through this action when $k = 2$ we needed Proposition 3.5, which states that the various maps $\mathcal{L}_1^T \rightarrow \mathcal{L}_3^T$ defined by $x \mapsto [x, z_j]$ have independent image. The corresponding statement for maps $\mathcal{L}_{k-1}^T \rightarrow \mathcal{L}_{k+1}^T$ is definitely false for $k > 2$: for example, we have relations such as $[z_i, z_j] + [z_j, z_i] = 0$ or $[[z_i, z_j], z_k] + [[z_j, z_k], z_i] + [[z_k, z_i], z_j] = 0$.

5.6 Naturality

Consider a morphism $\iota: \Sigma \rightarrow \Sigma'$ and the associated embedding $\widehat{i}: \widehat{\Sigma} \rightarrow \widehat{\Sigma}'$. In this section we seek to define a map

$$\iota_*: W_\Sigma \rightarrow W_{\Sigma'}$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}(\Sigma) & \xrightarrow{\tau_\Sigma} & W_\Sigma \\ \downarrow & & \downarrow \iota_* \\ \mathcal{I}(\Sigma') & \xrightarrow{\tau_{\Sigma'}} & W_{\Sigma'} \end{array} \quad (12)$$

As mentioned in the previous section, we may assume that $\Sigma = \widehat{\Sigma}$ and $\Sigma' = \widehat{\Sigma}'$. (Formally, if $\iota: \Sigma \rightarrow \widehat{\Sigma}$ is the natural inclusion, we define $\iota_*: W_\Sigma \rightarrow W_{\widehat{\Sigma}}$ to be the identity, and then the fact that (12) commutes is inherent in the definition of τ_Σ .)

We are now ready to define the map $\iota_*: W_{\Sigma_1} \rightarrow W_{\Sigma_2}$. We consider separately the case when ι is non-collapsing and the case when ι is a simple capping; since every morphism is a composition of such inclusions, this suffices. The description we give here is the one that is most useful in practice; in the proof we work with a more formal, but equivalent, definition.

Notational remark. It will be helpful to establish some simpler notation for elements of W_Σ . For the $\bigwedge^3 H(\Sigma)$ component, we have the standard $a \wedge b \wedge c$ notation. For the $H(\Sigma) \otimes \mathbb{Z}^k$ factor, in place of $\delta_i(f) \otimes z_i$ we write $\delta_i(f) \wedge z_i$, interpreting this as a formal expression.

Definition 5.11. For a simple capping $\iota: \Sigma \rightarrow \Sigma'$, so that Σ' is obtained from Σ by attaching a disk to the separating component z_i , the map ι_* sends $\delta_i(f) \wedge z_i \mapsto 0$ and is the identity on the other factors spanned by $\bigwedge^3 H(\Sigma)$ and by $\delta_j(f) \wedge z_j$ for $j \neq i$.

For a non-collapsing morphism $\iota: \Sigma \rightarrow \Sigma'$, decompose $\widehat{\Sigma}' \setminus \widehat{\Sigma}$ into subsurfaces U_i , so that U_i meets $\widehat{\Sigma}$ in the component z_i corresponding to P_i . We can consider $H(U_i)$ as a subspace of $H(\Sigma')$. Let $\omega_{U_i} \in \bigwedge^2 H(U_i)$ represent the intersection form on $H(U_i)$, and let z_i^1, \dots, z_i^l be the boundary components of U_i that are disjoint from $\widehat{\Sigma}$. We define ι_* as follows.

$$\begin{aligned} \iota_*: W_\Sigma &\rightarrow W_{\Sigma'} \\ a \wedge b \wedge c &\mapsto a \wedge b \wedge c \\ x \wedge z_i &\mapsto x \wedge (\omega_{U_i} + z_i^1 + \dots + z_i^l) \end{aligned}$$

Note that in $N(U_i)$ we have $z_i = \omega_{U_i} + z_i^1 + \dots + z_i^l$, since z_i is the boundary component that would be z_i^0 , but transversed in the opposite direction.

Theorem 5.12. For any inclusion $\iota: \Sigma_1 \rightarrow \Sigma_2$, with ι_* defined as in Definition 5.11, the diagram (12) commutes.

Proof. The argument is similar to the proof of Theorem 4.6. In the course of the proof, we will extend $\iota_*: W_\Sigma \rightarrow W_{\Sigma'}$ to a map $\iota'_*: \text{Hom}(H(\Sigma), N(\Sigma)) \rightarrow \text{Hom}(H(\Sigma'), N(\Sigma'))$. We can then use the original definition of τ_Σ to verify naturality for ι'_* , and it remains only to check that ι'_* does in fact restrict to ι_* .

First, assume that ι is a simple capping, so that Σ' is obtained from Σ by attaching a disk to the separating component z_j . Since z_j was separating and thus represented by $\zeta_j \in T(\Sigma)$, we have

$H(\Sigma') = H(\Sigma)$, and the natural map $N(\Sigma) \rightarrow N(\Sigma')$ is surjective with kernel generated by z_j . Thus the exact sequences defining τ_Σ and $\tau_{\Sigma'}$ are related by the following diagram (13):

$$\begin{array}{ccccccc}
& & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{Z} & & \mathbb{Z} & & \\
& & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N(\Sigma) & \longrightarrow & E(\Sigma) & \longrightarrow & H(\Sigma) \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \parallel \\
1 & \longrightarrow & N(\Sigma') & \longrightarrow & E(\Sigma') & \longrightarrow & H(\Sigma') \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \\
& & 1 & & 1 & &
\end{array} \tag{13}$$

It follows that $\tau_{\Sigma'}(\varphi) = \iota'_*(\tau_\Sigma(\varphi))$ for $\varphi \in \mathcal{I}(\Sigma)$, where ι'_* is the map

$$\iota'_*: \text{Hom}(H(\Sigma), N(\Sigma)) \rightarrow \text{Hom}(H(\Sigma'), N(\Sigma'))$$

sending $f \in \text{Hom}(H(\Sigma), N(\Sigma))$ to the composition

$$\iota'_* f: H(\Sigma') \cong H(\Sigma) \xrightarrow{f} N(\Sigma) \twoheadrightarrow N(\Sigma').$$

The restriction of ι'_* to W_Σ has kernel $H(\Sigma) \otimes z_j$ and coincides with ι_* as defined in Definition 5.11, verifying the theorem in this case.

Now, assume that ι is non-collapsing. In this case the induced map $\iota_*: N(\Sigma) \rightarrow N(\Sigma')$ is an injection by Theorem 3.2. Recall that the U_i are the components of $\widehat{\Sigma}' \setminus \widehat{\Sigma}$, labeled so that U_i intersects $\widehat{\Sigma}$ in the boundary component z_i . Each U_i inherits the structure of a partitioned surface from $\widehat{\Sigma}'$, and since the resulting partition is totally separated, the inclusion $U_i \rightarrow \widehat{\Sigma}'$ is uniquely determined. Identifying $H(\Sigma)$ with its image in $H(\Sigma')$, we have an orthogonal splitting $H(\Sigma') = H(\Sigma) \oplus \bigoplus_i H(U_i)$. We use this to define ι'_* as follows. Given a homomorphism $f \in \text{Hom}(H(\Sigma), N(\Sigma))$, let $\iota'_* f \in \text{Hom}(H(\Sigma'), N(\Sigma'))$ be the homomorphism defined by:

$$\iota'_* f(x) = \begin{cases} i_*(f(x)) & \text{if } x \in H(\Sigma) \\ \delta_i(f) \wedge x & \text{if } x \in H(U_i) \end{cases} \tag{14}$$

The term $\delta_i(f) \wedge x$ here is taken in $\bigwedge^2 H(\Sigma') \leq N(\Sigma')$. The proof that (12) commutes with this definition of ι'_* is exactly the same as the proof of Theorem 4.6. It remains to check that ι'_* restricts to W_Σ as in Definition 5.11. If $\{a_i^j, b_i^j\}$ form a symplectic basis for $H(U_i)$, we can rewrite Definition 5.11 as:

$$\delta_i(f) \otimes z_i \mapsto \delta_i(f) \otimes \left(\sum_j a_i^j \wedge b_i^j + \sum_k z_i^k \right) + \sum_j a_i^j \otimes b_i^j \wedge \delta_i(f) + b_i^j \otimes \delta_i(f) \wedge a_i^j \tag{15}$$

But we have already noted that $\omega_{U_i} + z_i^1 + \dots + z_i^l = z_i$ in $N(\Sigma')$, so (15) agrees with (14) on $H(\Sigma)$. The remaining summation corresponds to the homomorphism defined on $H(U_i)$ by $a_i^j \mapsto \delta_i(f) \wedge a_i^j, b_i^j \mapsto -b_i^j \wedge \delta_i(f)$, which is just $x \mapsto \delta_i(f) \wedge x$, so again (15) agrees with (14). \square

Moving the basepoint. Consider the case when Σ and Σ' are the same partitioned surface, differing only in that the basepoint of Σ' lies in $P_i \in \mathcal{P}$ instead of P_0 .

Lemma 5.13. *If Σ and Σ' coincide as partitioned surfaces except that $*' \in P_i$, then*

$$\tau_{\Sigma'}(\varphi) = \tau_{\Sigma}(\varphi) - d_i(\varphi) \wedge \omega.$$

Proof. Recall that an isomorphism $\widehat{\pi} \rightarrow \widehat{\pi}'$ is given by $x \mapsto \bar{x} = A_i^{-1}x A_i$. We saw in (5) that $\varphi(\bar{x})\bar{x}^{-1} = [d_i(\varphi)^{-1}, \varphi(x)]\varphi(x)x^{-1}$. Comparing $\tau_{\Sigma}(\varphi) \in \text{Hom}(H(\Sigma), N(\Sigma))$ with $\tau_{\Sigma'} \in \text{Hom}(H(\Sigma'), N(\Sigma'))$ along the induced isomorphisms $H(\Sigma) \cong H(\Sigma')$ and $N(\Sigma) \cong N(\Sigma')$, this gives that that

$$\tau_{\Sigma'}(\varphi)(x) = \tau_{\Sigma}(\varphi)(x) - d_i(\varphi) \wedge x.$$

The homomorphism $x \mapsto -d_i(\varphi) \wedge x$ in $\text{Hom}(H(\Sigma), \bigwedge^2 H(\Sigma))$ is represented by $-d_i(\varphi) \otimes \omega \in H(\Sigma) \otimes \bigwedge^2 H(\Sigma)$, verifying the lemma. \square

Viewing τ as a natural transformation. We can sum up the properties of τ_{Σ} by interpreting it as a natural transformation as follows. We have already noted in Section 2.3 that \mathcal{I} can be considered as a functor from $\mathcal{T}\text{Surf}$ to AbGrp (the category of abelian groups and homomorphisms), defined on objects by $\Sigma \mapsto \mathcal{I}(\Sigma)$ and on morphisms by $\iota \mapsto \iota_*$. Let \mathcal{W} be the functor from $\mathcal{T}\text{Surf}$ to AbGrp defined by $\mathcal{W}(\Sigma) = W_{\Sigma}$ and $\mathcal{W}(\iota) = \iota_*$ as in Definition 5.11. Then we can rephrase Theorem 5.12 as follows:

Theorem 5.14. *Assigning to each surface Σ the surjective homomorphism $\tau_{\Sigma}: \mathcal{I}(\Sigma) \rightarrow W_{\Sigma}$ gives a natural transformation from the Torelli functor \mathcal{I} to the functor \mathcal{W} .*

6 Fundamental calculations and surjectivity of τ_{Σ}

We calculate τ_{Σ} in a number of fundamental situations, including the natural “point-pushing” subgroups. Using this, we prove in Section 6.5 that W_{Σ} is exactly the image of τ_{Σ} . These results are used in Section 7.

6.1 Separating twists

Let Σ be the surface $S_{0,n}$ with the totally separated partition. The homology group $H(\Sigma)$ is trivial. It follows that the range W_{Σ} of τ_{Σ} is trivial, so that $\tau_{\Sigma}(\varphi) = 0$ for any $\varphi \in \mathcal{I}(\Sigma)$. Applying the naturality of τ_{Σ} , we obtain the following corollaries, which apply to any partitioned surface.

Proposition 6.1. *For any separating twist T_{γ} , $\tau_{\Sigma}(T_{\gamma}) = 0$.*

This is because any Dehn twist T_{γ} is supported on the annulus which is the regular neighborhood of γ , and this annulus $S_{0,2}$ has the totally separated partition exactly if γ is a separating curve. More generally, we have the following.

Proposition 6.2. *If $\varphi \in \mathcal{I}(\Sigma)$ is supported on a totally separated genus 0 subsurface, then $\tau_{\Sigma}(\varphi) = 0$.*

6.2 Bounding pair maps

Let Σ be a surface $S_{0,3}$ of genus 0 with 3 boundary components z_0 , z_1 , and z_2 , endowed with the partition $\mathcal{P} = \{\{z_0\}, \{a_1, a_2\}\}$ and basepoint $* \in z_0$. Let $\varphi = T_{a_1}T_{a_2}^{-1}$. This is a bounding pair, and Σ is the minimal connected surface on which φ is supported. The surface $\widehat{\Sigma}$ has genus 1 with 2 boundary components, z_0 and z_1 . Its fundamental group has rank 3, and we may take the basis $\{\alpha, \beta, \zeta\}$ so that the first two terms descend to a basis $\{a, b\}$ for $H(\Sigma)$, and ζ descends to $z = z_1$ in $N(\Sigma)$. For an appropriate choice of generators we have $\varphi(\alpha) = \alpha$ and $\varphi(\beta) = \beta\zeta^{-1}$, so $\tau_\Sigma(\varphi)$ is defined by $a \mapsto 0$ and $b \mapsto -z$. In the notation of Section 5.6, we have $\tau_\Sigma(\varphi) = a \wedge z$ in W_Σ . The same argument applies when the basepoint lies in z_1 . Since every bounding pair in a partitioned surface Σ sits inside at least one such $S_{0,3}$ (for example, the regular neighborhood of the curves together with an arc connecting them), we can apply the naturality of τ_Σ to obtain the following corollary.

Proposition 6.3. *Given a bounding pair $T_\gamma T_\delta^{-1}$ defined by nonseparating curves γ and δ , let ζ be an separating curve which cobounds a pair of pants with $\gamma \cup \delta$ as oriented curves.² Let a be the homology class of γ , and let z be the class of ζ in $N(\Sigma)$. Then we have $\tau_\Sigma(T_\gamma T_\delta^{-1}) = a \wedge z$ in W_Σ .*

Note that changing the orientation of all three curves negates a and z , and thus preserves $a \wedge z$. Similarly, choosing ζ on the other side of $\gamma \cup \delta$ changes z by a term of the form $a \wedge b$, so $a \wedge z$ is unchanged.

6.3 Lantern core maps

We say that α and β span a *lantern core* if their geometric intersection number is 2 and their algebraic intersection number is 0. A *lantern core map* is the commutator $[T_\alpha, T_\beta]$ of Dehn twists around two such curves. Since α and β have algebraic intersection 0, T_α and T_β act on $H(\Sigma)$ by commuting transvections. It follows that $[T_\alpha, T_\beta] \in \mathcal{I}(\Sigma)$. Such maps were first used by Johnson in [7]; they are called simply intersecting pair maps in Putman [10].

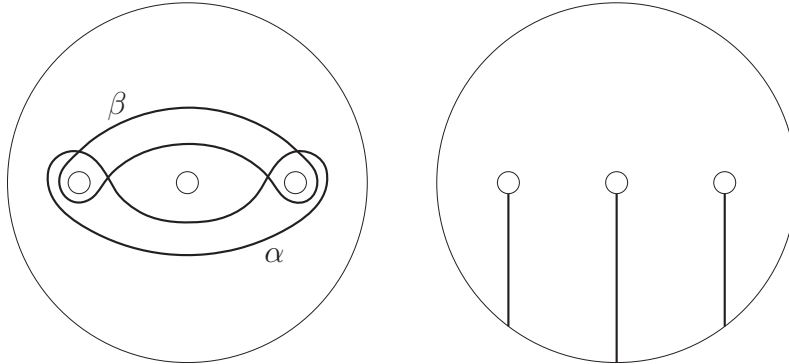


Figure 1: The simple closed curves α and β , and the arcs B_1, B_2, B_3 .

The regular neighborhood of two such curves is always a lantern, so it suffices to compute τ_Σ for Σ homeomorphic to $S_{0,4}$. To begin, we say that α and β span a *nonseparating lantern core* if $\alpha \cup \beta$ is \mathcal{P} -nonseparating; that is, the induced partition on the regular neighborhood $S_{0,4}$ of $\alpha \cup \beta$ is the

²This means that the curves are oriented so that the pair of pants lies on the left side of ζ and the right side of γ and δ , or vice versa.

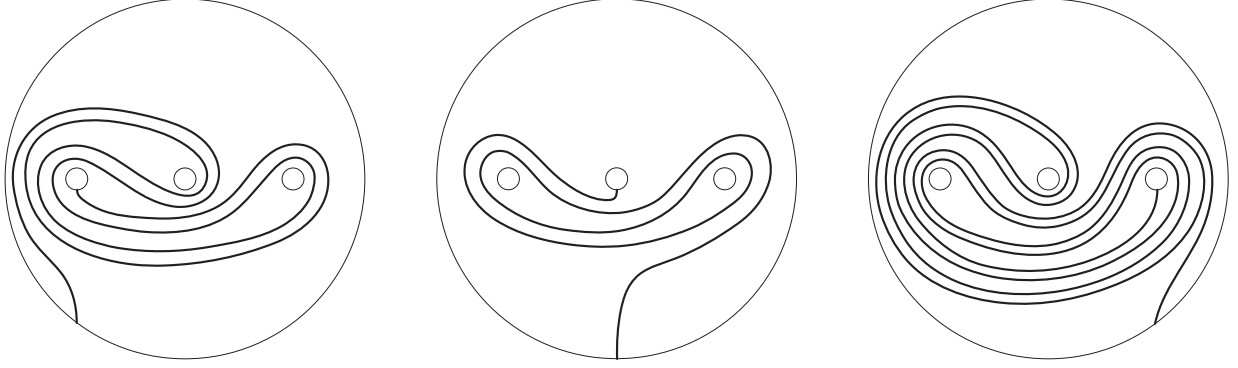


Figure 2: The arcs $\varphi(B_1)$, $\varphi(B_2)$ and $\varphi(B_3)$.

nonseparating partition. Let α and β be as in Figure 6.3, and let a_1, a_2, a_3 be the homology classes in $H(\Sigma)$ of the three boundary components in the center.

Proposition 6.4. *If α and β span a nonseparating lantern core, then $\tau_\Sigma([T_\alpha, T_\beta]) = -a_1 \wedge a_2 \wedge a_3$.*

Proof. The surface $\widehat{\Sigma}$ has genus 3 with 1 boundary component. A basis for $\widehat{\pi}$ is given by curves $\alpha_1, \alpha_2, \alpha_3$ traveling clockwise the three central boundary components, together with curves $\beta_1, \beta_2, \beta_3$ whose intersection with Σ are the arcs B_1, B_2, B_3 respectively, oriented bottom-to-top. These descend to a basis $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ for $H(\Sigma)$.

Let $\varphi = [T_\alpha, T_\beta]$. Since a_1, a_2 , and a_3 are contained in D_0 , we have

$$\tau_\Sigma(\varphi)(a_1) = \tau_\Sigma(\varphi)(a_2) = \tau_\Sigma(\varphi)(a_3) = 0.$$

The action of φ on the arcs B_i is displayed in Figure 6.3. Thus we have:

$$\begin{aligned} B_1 &\mapsto [\alpha_1 \alpha_2 \alpha_1^{-1}, \alpha_3] B_1 \\ B_2 &\mapsto [\alpha_3^{-1}, \alpha_1^{-1}] B_2 \\ B_3 &\mapsto [\alpha_3^{-1} \alpha_1^{-1} \alpha_3, \alpha_1 \alpha_2^{-1} \alpha_1^{-1}] B_3 \end{aligned} \tag{16}$$

It follows that:

$$\begin{aligned} \tau_\Sigma(\varphi)(b_1) &= a_2 \wedge a_3 \\ \tau_\Sigma(\varphi)(b_2) &= (-a_3) \wedge (-a_1) = a_3 \wedge a_1 \\ \tau_\Sigma(\varphi)(b_3) &= (-a_1) \wedge (-a_2) = a_1 \wedge a_2 \end{aligned}$$

Thus as an element of $W_\Sigma = \bigwedge^3 H(\Sigma)$, we have $\tau_\Sigma(\varphi) = -a_1 \wedge a_2 \wedge a_3$, as desired. The naturality of τ_Σ implies that the same formula holds for a nonseparating lantern core in any surface Σ . \square

From the same computations we can deduce that any other lantern core map is contained in the Johnson kernel $\mathcal{K}(\Sigma)$. Indeed for any other surface $\Sigma = (S_{0,4}, \mathcal{P}, *)$ for which \mathcal{P} is not the nonseparating partition, the rank of $D(\Sigma)^\perp$ is at most 2. Thus $\bigwedge^3 D(\Sigma)^\perp = 0$, so by Remark 5.8, $\tau_\Sigma(\varphi)$ is determined by the $d_i(\varphi) \in H(\Sigma)$. But we saw in (16) that $\varphi(B_i)B_i^{-1} \in [\widehat{\pi}, \widehat{\pi}]$ for each i , and so $d_i(\varphi) = 0$ for all i .

Corollary 6.5. *If α and β span a lantern core which is not nonseparating, then $\tau_\Sigma([T_\alpha, T_\beta]) = 0$.*

6.4 Disk-pushing subgroups

In this section we determine the restriction of τ_Σ to certain “disk-pushing” subgroups of $\mathcal{I}(\Sigma)$. Let $\Sigma \rightarrow \Sigma'$ be a simple capping obtained by attaching a disk to a separating boundary component $z = z_i$. In particular, we assume that Σ has at least two boundary components, and that $*$ is not contained in z . Such a capping induces a surjection $\text{Mod}(\Sigma) \rightarrow \text{Mod}(\Sigma')$, whose kernel is isomorphic to $\pi_1(UTS', v)$, where v is a unit vector at the center of the disk glued to z (Johnson [6]).

Remark 6.6. The conventions for composition in $\text{Mod}(\Sigma)$ and in $\pi_1(UTS', v)$ unfortunately disagree; in the former we take composition of functions and in the latter we take concatenation of paths. As a result, the isomorphism of $\ker(\text{Mod}(\Sigma) \rightarrow \text{Mod}(\Sigma'))$ with $\pi_1(UTS', v)$ is defined as follows. Given $\varphi \in \ker(\text{Mod}(\Sigma) \rightarrow \text{Mod}(\Sigma'))$, extend φ by the identity to S' ; by definition, there is an isotopy h_t^φ of S' from $h_0^\varphi = \varphi$ to $h_1^\varphi = \text{id}$. Since φ fixes v , the image of v under this isotopy is a loop of tangent vectors, and we associate to φ this loop $\gamma_\varphi := h_t^\varphi(v) \in \pi_1(UTS', v)$. Note that given another map ψ fixing v , $h_t^\varphi \circ \psi$ satisfies $h_t^\varphi \circ \psi(v) = h_t^\varphi(v)$. Thus an isotopy from $\varphi \circ \psi$ to id can be obtained by concatenating the isotopy $h_t^\varphi \circ \psi$ from $\varphi \circ \psi$ to ψ with the isotopy h_t^ψ from ψ to id ; the path this determines is the concatenation $\gamma_\varphi \cdot \gamma_\psi$. This verifies that $\gamma_{\varphi \circ \psi} = \gamma_\varphi \cdot \gamma_\psi$. We remark that this isomorphism is the opposite of the identification naively suggested by the “disk-pushing” label.

Proposition 6.7. *For a separating component, the entire disk-pushing subgroup $\pi_1(UTS', v)$ is contained in $\mathcal{I}(\Sigma)$.*

Proof. Note that we have $H(\Sigma') \simeq H(\Sigma)$, since z is separating and thus vanishes in homology. For any curve C in S and any $\varphi \in \pi_1(UTS', v)$, the curve $\varphi(C)$ is homotopic to C in S' , which implies that $\varphi(C)$ is homologous to C in $H(\Sigma)$. Thus $\varphi \in \mathcal{I}(\Sigma)$ as desired. \square

We may thus consider the restriction $\tau_\Sigma: \pi_1(UTS', v) \rightarrow W_\Sigma$.

Proposition 6.8. *The restriction of τ_Σ to the disk-pushing subgroup determined by z_i is the composition*

$$\ker(\mathcal{I}(\Sigma) \rightarrow \mathcal{I}(\Sigma')) \approx \pi_1(UTS', d) \twoheadrightarrow \pi_1(S', d) \rightarrow H(\Sigma') \approx H(\Sigma) \hookrightarrow W_\Sigma,$$

where the last map is the embedding $H(\Sigma) \hookrightarrow W_\Sigma$ defined by $x \mapsto -x \wedge z_i$.

Let $d \in S'$ be the projection of $v \in UTS'$. Given $\gamma \in \pi_1(UTS', v)$, let $\bar{\gamma}$ be its projection to $\pi_1(S', d)$.

Proof. For any $\varphi \in \ker(\mathcal{I}(\Sigma) \rightarrow \mathcal{I}(\Sigma'))$, the naturality of τ_Σ implies that $\tau_\Sigma(\varphi) \in \ker(W_\Sigma \rightarrow W_{\Sigma'})$. For a simple capping $\Sigma \rightarrow \Sigma'$, by Definition 5.11 the map $W_\Sigma \rightarrow W_{\Sigma'}$ is induced by the quotient $N(\Sigma) \twoheadrightarrow N(\Sigma)/\langle z_i \rangle = N(\Sigma')$. We thus know *a priori* that $\tau_\Sigma(\varphi)$ must be contained in the subspace $\{x \wedge z_i | x \in H(\Sigma)\}$. By Theorem 5.5 we thus have

$$\tau_\Sigma(\varphi) = d_i(\varphi) \wedge z_i.$$

Thus it suffices to prove that if $\varphi \in \ker(\mathcal{I}(\Sigma) \rightarrow \mathcal{I}(\Sigma'))$ corresponds to $\gamma_\varphi \in \pi_1(UTS', v)$, we have

$$d_i(\varphi) = -[\bar{\gamma}_\varphi]. \tag{17}$$

Recall that $d_i(\varphi)$ is the homology class of $\varphi(A)A^{-1}$ where $A = A_i$ is an arc from the basepoint to z_i . Since $H(\Sigma) \approx H(\Sigma')$, we may consider A as an arc in S' from the basepoint to d . If $\bar{\gamma}_\varphi$ is simple and disjoint from A as a based loop, then $\varphi(A)$ will be homotopic in S' to the concatenation $A \cdot \bar{\gamma}_\varphi^{-1}$.

Note that if $\overline{\gamma_\varphi}$ is simple, we can always choose A disjoint from it. For such elements, we have $\varphi(A)A^{-1} = A \cdot \overline{\gamma_\varphi}^{-1} \cdot A^{-1}$. This is homologous in S' to $[\overline{\gamma_\varphi}^{-1}] = -[\overline{\gamma_\varphi}]$, and since $H(\Sigma') \simeq H(\Sigma)$ this implies that $[\varphi(A)A^{-1}] = -[\overline{\gamma_\varphi}] \in H(\Sigma)$. Thus for such elements (17) holds and $\tau_\Sigma(\varphi) = -[\overline{\gamma_\varphi}] \wedge z$.

The kernel $\ker(\pi_1(UTS', v) \rightarrow \pi_1(S', d))$ is generated by a twist around the boundary component z_i . By Proposition 6.1 τ_Σ vanishes for any separating twist, so (17) holds for this element as well. The group $\pi_1(S', d)$ is normally generated by (in fact, generated by) elements represented by simple loops. It follows that $\pi_1(UTS', v)$ is normally generated by γ for which $\overline{\gamma}$ is either simple or trivial. Since (17) holds for generators of either form, it holds for all elements of $\ker(\mathcal{I}(\Sigma) \rightarrow \mathcal{I}(\Sigma')) \approx \pi_1(UTS', v)$. This completes the proof of the proposition. \square

Remark 6.9. Note that for many surfaces Σ , the subgroups $\pi_1(UTS', v)$ is generated by bounding pairs, so we could obtain another proof of the proposition by applying Proposition 6.3. This was the approach used by Johnson to prove the proposition in the classical case when S has a single boundary component. In fact the proposition is almost automatic in this case: the restriction of τ_S to the point-pushing subgroup $\pi_1(UTS', v)$ is a map to the torsion-free abelian group $\bigwedge^3 H_1(S; \mathbb{Z})$, so it factors through the surjection $\pi_1(UTS', v) \twoheadrightarrow H_1(S; \mathbb{Z})$, and moreover extends to a map $H_1(S; \mathbb{Q}) \rightarrow \bigwedge^3 H_1(S; \mathbb{Q})$. Since $H_1(S; \mathbb{Q})$ is an irreducible $\mathrm{Sp}_{2g} \mathbb{Z}$ -module and $\bigwedge^3 H_1(S; \mathbb{Q})$ contains a unique submodule isomorphic to $H_1(S; \mathbb{Q})$ (embedded by $x \mapsto x \wedge \omega$), Schur's lemma implies that the proposition holds up to a multiplicative constant. This constant can be detected by computing τ_S for a single element.

6.5 Surjectivity of τ_Σ

To complete the proof of Theorem 5.7, it remains to show that $\mathrm{im} \tau_\Sigma$ is all of W_Σ . Recall that the projection $W_\Sigma \rightarrow (D(\Sigma)^\perp)^k$ defined by $f \mapsto (\delta_i(f), \dots, \delta_k(f))$ fits into an exact sequence:

$$0 \rightarrow \bigwedge^3 D(\Sigma)^\perp \rightarrow W_\Sigma \rightarrow (D(\Sigma)^\perp)^k \rightarrow 0$$

Fix a symplectic basis $\{a_i, b_i\} \cup \{a_j^i, b_j^i\}$ for $H(\Sigma)$ so that $\{a_j^i\}$ provides a basis for D_j , each a_j^i is represented by a boundary component, and $\{a_i, b_i\} \cup \{a_j^i\}$ provides a basis for $D(\Sigma)^\perp$.

The image $\mathrm{im} \tau_\Sigma$ surjects to $(D(\Sigma)^\perp)^k$. Fix $J \geq 1$ and let η_J be the \mathcal{P} -separating curve cutting off a subsurface of genus 0 bounded by η_J together with all the boundary components lying in P_J . We will show that for any $x \in \{a_i, b_i\} \cup \{a_j^i\}$ that we can find $\varphi \in \mathcal{I}(\Sigma)$ with $\tau_\Sigma(\varphi) = x \wedge \eta_J$. Note that $\eta_J = [a_J^1, b_J^1] + \dots + [a_J^{m_J}, b_J^{m_J}] + z_J$ in $N(\Sigma)$, so $f = x \wedge \eta_J$ has $\delta_J(f) = x$ and $\delta_j(f) = 0$ for $j \neq J$. Since these x span $D(\Sigma)^\perp$, this will verify that $\mathrm{im} \tau_\Sigma$ surjects to $(D(\Sigma)^\perp)^k$.

First, consider the case when $x = a_J^i$. Let γ be a curve homotopic to a_J^i , and let δ be the band sum of γ with η_J . The homologous curves γ and δ determine a bounding pair $T_\gamma T_\delta^{-1}$ and cobound a pair of pants with η_J , so by Proposition 6.3 we have $\tau_\Sigma(T_\gamma T_\delta^{-1}) = a_J^i \wedge \eta_J$.

For the remaining cases, let S_J be the component of $S - \eta_J$ containing the basepoint, and consider the disk-pushing subgroup of $\mathcal{I}(\Sigma_J)$ determined by η_J . By Proposition 6.8, we can find $\varphi \in \mathcal{I}(\Sigma_J)$ with $\tau_{\Sigma_J}(\varphi) = x \wedge \eta_J$ for any $x \in H(\Sigma_J)$ in the image of $\pi_1(S_J)$. It is easy to realize x by a loop in S_J whenever x is any a_i , any b_i , or any a_j^i with $j \neq J$. Applying Theorem 5.12 to the non-collapsing inclusion of Σ_J into Σ , this yields $\varphi \in \mathcal{I}(\Sigma)$ with $\tau_\Sigma(\varphi) = x \wedge \eta_J$ for such x , as desired.

The image $\mathrm{im} \tau_\Sigma$ contains $\bigwedge^3 D(\Sigma)^\perp$. Consider the natural basis of $\bigwedge^3 D(\Sigma)^\perp$ induced by the basis $\{a_i, b_i\} \cup \{a_j^i\}$ of $D(\Sigma)^\perp$. First, consider the basis elements of the form $x \wedge a_I \wedge b_I$ for some I and some other basis element x . Realize a_I and b_I by simple closed curves intersecting once, and let

η_I be the regular neighborhood of their union. Let S_I be the component of $S - \eta_I$ containing the basepoint, and consider the disk-pushing subgroup of $\mathcal{I}(\Sigma_I)$ determined by η_I . Any x in our basis distinct from a_I and b_I can be realized by a loop in S_I , so by Proposition 6.8 we can find $\varphi \in \mathcal{I}(\Sigma_I)$ with $\tau_{\Sigma_I}(\varphi) = x \wedge \eta_I$. Since $\eta_I = a_I \wedge b_I$ in $N(\Sigma)$, applying Theorem 5.12 to the non-collapsing inclusion of Σ_I into Σ implies that $\tau_{\Sigma}(\varphi) = x \wedge a_I \wedge b_I$.

Any other basis element is of the form $x \wedge y \wedge z$ where $(x, y) = (y, z) = (z, x) = 0$. We may thus realize these homology classes by disjoint curves α, β, γ . Let δ be the band sum of all three curves (to be precise we should specify orientations, but this will only change the answer by a sign). We obtain a lantern $S_{0,4}$ bounded by α, β, γ , and δ . If φ is a lantern core map supported on this lantern, then by Proposition 6.4, $\tau_{\Sigma}(\varphi) = \pm x \wedge y \wedge z$. This verifies that a basis for $\bigwedge^3 D(\Sigma)^{\perp}$ is contained in $\text{im } \tau_{\Sigma}$. Together with Theorem 5.9, this completes the proof of Theorem 5.7 and shows that the image of τ_{Σ} is exactly W_{Σ} . \square

7 Orbits of curves under $\mathcal{K}(\Sigma)$

7.1 $\mathcal{K}(\Sigma)$ -orbits, abstract description

In this section, we give an abstract description of the space of $\mathcal{K}(\Sigma)$ -orbits of a multicurve; this allows us to give concrete invariants in the sections that follow. A multicurve $\bar{\gamma}$ is just a collection of disjoint simple closed curves. Note that if multicurves $\bar{\gamma}$ and $\bar{\delta}$ are to be in the same $\mathcal{K}(\Sigma)$ -orbit, they certainly must be in the same $\mathcal{I}(\Sigma)$ -orbit.

Theorem 7.1. *For any multicurve $\bar{\gamma}$ on Σ , the space of $\mathcal{K}(\Sigma)$ -orbits within the $\mathcal{I}(\Sigma)$ -orbit of $\bar{\gamma}$ is $\mathcal{I}(\Sigma)$ -equivariantly isomorphic to an affine space modeled on the abelian group $W_{\Sigma}/\tau_{\Sigma}(\text{Stab}(\bar{\gamma}))$.*

Recall that an affine space A modeled on an abelian group G is a set A endowed with a relative invariant $d(a, a') \in G$ satisfying:

- $d(a, a') + d(a', a'') = d(a, a'')$
- $d(a, a') = 0 \iff a = a'$
- d is surjective

Proof. Given two collections of curves $\bar{\delta}$ and $\bar{\delta}'$ lying in the $\mathcal{I}(\Sigma)$ -orbit of $\bar{\gamma}$, choose $\varphi \in \mathcal{I}(\Sigma)$ so that $\varphi(\bar{\delta}) = \bar{\delta}'$. Then the relative invariant is defined by

$$d_{\bar{\gamma}}(\bar{\delta}, \bar{\delta}') = [\tau_{\Sigma}(\varphi)] \in W_{\Sigma}/\tau_{\Sigma}(\text{Stab}(\bar{\gamma})). \quad (18)$$

First, we check that this is well-defined. If we had chosen another $\varphi' \in \mathcal{I}(\Sigma)$ with $\varphi'(\bar{\delta}) = \bar{\delta}'$, we would have $\varphi^{-1}\varphi'(\bar{\delta}) = \bar{\delta}$, so $\varphi^{-1}\varphi' \in \text{Stab}(\bar{\delta})$. This is conjugate to $\text{Stab}(\bar{\gamma})$, so since τ_{Σ} is a map to an abelian group we have $\tau_{\Sigma}(\varphi^{-1}\varphi') \in \tau_{\Sigma}(\text{Stab}(\bar{\gamma}))$. It follows that $[\tau_{\Sigma}(\varphi)] = [\tau_{\Sigma}(\varphi')] \in W_{\Sigma}/\tau_{\Sigma}(\text{Stab}(\bar{\gamma}))$, as desired. The equivariance follows similarly: if $\varphi(\bar{\delta}) = \bar{\delta}'$, then we can compute

$$d_{\bar{\gamma}}(\psi(\bar{\delta}), \psi(\bar{\delta}')) = [\tau_{\Sigma}(\psi \cdot \varphi \cdot \psi^{-1})] = [\tau_{\Sigma}(\varphi)] = d_{\bar{\gamma}}(\bar{\delta}, \bar{\delta}').$$

Finally if $d_{\bar{\gamma}}(\bar{\delta}, \bar{\delta}') = 0$, then choosing φ for which $\varphi(\bar{\delta}) = \bar{\delta}'$ we have $[\tau_{\Sigma}(\varphi)] = 0$, so $\tau_{\Sigma}(\varphi) \in \tau_{\Sigma}(\text{Stab}(\bar{\gamma})) = \tau_{\Sigma}(\text{Stab}(\bar{\delta}))$. We may thus choose $\psi \in \text{Stab}(\bar{\delta})$ with $\tau_{\Sigma}(\psi) = \tau_{\Sigma}(\varphi)$. Then $\varphi\psi^{-1}$ satisfies $\varphi\psi^{-1}(\bar{\delta}) = \varphi(\bar{\delta}) = \bar{\delta}'$, and also $\tau_{\Sigma}(\varphi\psi^{-1}) = 0$, verifying that $\bar{\delta}$ and $\bar{\delta}'$ lie in the same $\mathcal{K}(\Sigma)$ -orbit. \square

In the remainder of Section 7 we give an explicit and computable version of this identification for the most useful types of configurations of curves.

7.2 Image of the symplectic representation

In this section we describe the image of the mapping class group $\text{Mod}(\Sigma)$ under the symplectic representation. This generalizes results proved in Johnson [5], and will be used in Sections 7.3 and 7.4.

As described in Section 2.2, $\text{Mod}(\Sigma)$ naturally acts on $H(\Sigma)$, yielding the so-called symplectic representation $\Psi: \text{Mod}(\Sigma) \rightarrow \text{Aut}(H(\Sigma))$. In this paper we only need to compute the image of Ψ in $\text{Aut}(H(\Sigma))$. However, the proof also determines the image of the action of $\text{Mod}(\Sigma)$ on $H(\Sigma) \otimes \mathbb{Z}/L\mathbb{Z}$ for any $L \in \mathbb{Z}$. We abbreviate this by $H(\Sigma; L) := H(\Sigma) \otimes \mathbb{Z}/L\mathbb{Z}$, and denote the representation $\Psi: \text{Mod}(\Sigma) \rightarrow \text{Aut}(H(\Sigma; L))$ by Ψ as well. We often identify \mathbb{Z} with $\mathbb{Z}/0\mathbb{Z}$ and use the notation $H(\Sigma; L)$ for both cases. As above $D = D(\Sigma) \leq H(\Sigma)$ is the isotropic subspace spanned by the homology classes of the boundary components; we denote its image in $H(\Sigma; L)$ by D as well.

Lemma 7.2. *The image of $\text{Mod}(\Sigma)$ under the symplectic representation is $\text{Sp}(H(\Sigma), D)$, those automorphisms of $H(\Sigma)$ preserving the intersection form and fixing the subspace D . Similarly the image of $\text{Mod}(\Sigma)$ in $\text{Aut}(H(\Sigma; L))$ is $\text{Sp}(H(\Sigma; L), D)$.*

Proof. Since $\text{Mod}(\Sigma)$ preserves the symplectic form on $H(\Sigma; L)$ and fixes the boundary components representing D , $\Psi(\text{Mod}(\Sigma))$ is clearly contained in $\text{Sp}(H(\Sigma; L), D)$. It remains to show that the image of $\text{Mod}(\Sigma)$ is all of $\text{Sp}(H(\Sigma; L), D)$.

We proceed by induction on $\text{rank } D = |\pi_0(\partial S)| - |\mathcal{P}|$. The base case is when \mathcal{P} is totally separated and D is trivial. In this case the map $\text{Mod}(\Sigma) \rightarrow \text{Mod}(\bar{S})$ is surjective, and it is a classical fact that $\Psi(\text{Mod}(\bar{S})) = \text{Sp}(H_1(\bar{S})) = \text{Sp}(H(\Sigma))$. The natural map $\text{Sp}(H(\Sigma)) \rightarrow \text{Sp}(H(\Sigma; L))$ is also surjective for any L . This fact must be well-known; for one proof, Hahn–O’Meara [3, Theorems 4.3.9 and 9.2.5] show that $\text{Sp}(H(\Sigma; L))$ is generated by elementary matrices (since $\mathbb{Z}/L\mathbb{Z}$ is a semilocal ring), and elementary matrices can be lifted to $\text{Sp}(H(\Sigma))$.

For the inductive step, some component $P \in \mathcal{P}$ is nontrivial; choose two components z_0, z_1 from P , and let $a \in H(\Sigma; L)$ be the homology class of z_1 . Let γ be a curve cutting S into two components, the first component a pair of pants bounded by z_0, z_1 , and γ , and the second component a subsurface S' of S . Choose a curve β contained in $\bar{S} \setminus S'$ intersecting z_0 once and z_1 once, with homology class $b \in H(\Sigma)$. The key is that we now have a decomposition

$$H(\Sigma; L) = \langle a, b \rangle \oplus H(\Sigma'; L)$$

where both factors $\langle a, b \rangle$ and $H(\Sigma'; L)$ are symplectic subspaces, and since a and b are represented by curves disjoint from $H(\Sigma'; L)$, this decomposition is orthogonal. Note that if $D' = D(\Sigma') \leq H(\Sigma'; L)$ is the isotropic subspace spanned by boundary components of S' , we have that $D' = D \cap H(\Sigma'; L)$.

It now suffices to show that for any $\psi \in \text{Sp}(H(\Sigma; L), D)$, we can find $\varphi \in \text{Mod}(\Sigma)$ so that $\varphi(b) = \psi(b)$. This is proved below, but first we complete the argument. Consider the automorphism $\varphi^{-1}\psi$; it lies in $\text{Sp}(H(\Sigma; L), D)$ and fixes a and b , so we may restrict it to the complement $H(\Sigma'; L)$. Its restriction lies in $\text{Sp}(H(\Sigma'; L), D')$. Applying the inductive hypothesis to Σ' , there is some $\varphi' \in \text{Mod}(\Sigma')$ inducing $\varphi^{-1}\psi$ on $H(\Sigma'; L)$; it follows that $\varphi\varphi' \in \text{Mod}(\Sigma)$ induces ψ , as desired.

It remains to find $\varphi \in \text{Mod}(\Sigma)$ with $\varphi(b) = \psi(b)$. For each $P \in \mathcal{P}$, choose a separating curve bounding a genus 0 subsurface with all the components in P . Let S_0 be the core subsurface cut off by these curves; note that \mathcal{P}_0 is totally separated, and that $H(\Sigma_0; L)$ is a symplectic subspace of $H(\Sigma; L)$. We can extend $\{a\}$ to a basis $\{a, a_2, \dots, a_k\}$ for D such that each a_i is represented by a boundary component z_i with the orientation induced by S , and $(b, a_i) = 0$ for all $i \geq 2$. The fact that ψ is in $\text{Sp}(H(\Sigma; L), D)$ implies that $\psi(b) = b + ma + \sum m_i a_i + c$ for some $c \in H(\Sigma_0; L)$ and $m, m_i \in \mathbb{Z}$. We find a series of Dehn twists taking $b \mapsto b + ma + \sum m_i a_i + c$; we remark that these twists will commute in their action on b .

If $L = 0$, we can write $c = nc_0$ for some primitive vector $c_0 \in H(\Sigma_0)$; if $L \neq 0$, lift $c \in H(\Sigma_0; L)$ to $\tilde{c} \in H(\Sigma_0)$ and take $\tilde{c} = nc_0$. By Meeks–Patrusky [8], c_0 can be realized by a simple closed curve in S_0 , and by plumbing this curve with z_1 along an appropriate arc, we obtain a simple closed curve γ_{a+c_0} representing $a+c_0$ in $H(\Sigma; L)$. Since $(b, a+c_0) = 1$, we have that $T_{\gamma_{a+c_0}}^n(b) = b+na+nc_0 = b+na+c$ in $H(\Sigma; L)$. By plumbing z_1 and z_i along an arc, we obtain a simple closed curve γ_{a+a_i} representing $a+a_i$ (here we use the condition on the orientations of the a_i). Similarly we have $(b, a+a_i) = 1$ and thus $T_{\gamma_{a+a_i}}^{m_i}(b) = b+m_i a+m_i a_i$. Finally, note that $T_{z_1}^m(b) = b+ma$. Note that, as mentioned above, all these twists commute in their action on b . By direct calculation we have:

$$T_{z_1}^{m-n-\sum m_i} \circ T_{\gamma_{a+a_k}}^{m_k} \circ \dots \circ T_{\gamma_{a+a_2}}^{m_2} \circ T_{\gamma_{a+c_0}}^n(b) = b+ma + \sum m_i a_i + c = \psi(b). \quad \square$$

We remark that this lemma holds even when the genus of Σ is 0 or 1; in the former case D is a maximal isotropic subspace of $H(\Sigma)$ and $\text{Sp}(H(\Sigma), D)$ is isomorphic to the abelian group $\text{Sym}^2 D$.

We have the following corollary, which applies to the subspace of homology supported on S and is used in the next section. This corollary generalizes Lemma 5 of Johnson [5]. As before, let H_Σ be the subspace of $H(\Sigma; L)$ spanned by $H_1(S)$ and let $D \leq H_\Sigma$ be the subspace spanned by boundary components. H_Σ is a primitive subspace of $H(\Sigma; L)$ and inherits an intersection form, which is no longer nondegenerate; we have

$$D = \text{rad}(H_\Sigma) = \{x \in H_\Sigma \mid \forall y \in H_\Sigma : (x, y) = 0\}.$$

Corollary 7.3. *The image of $\text{Mod}(\Sigma)$ in $\text{Aut}(H_\Sigma)$ is $\text{Sp}(H_\Sigma, D)$, those automorphisms of H_Σ preserving the intersection form and fixing the subspace D .*

Proof. Deriving this corollary from Lemma 7.2 is a matter of linear algebra. Note first that $H_\Sigma = D^\perp$, the orthogonal complement of D in $H(\Sigma; L)$. Thus H_Σ is preserved by $\text{Sp}(H(\Sigma; L), D)$, yielding a restriction map $\text{Sp}(H(\Sigma; L), D) \rightarrow \text{Aut}(H_\Sigma)$ whose image is contained in $\text{Sp}(H_\Sigma, D)$; the claim is that this map is surjective. This is equivalent to the general fact that any automorphism ψ of H_Σ fixing D may be extended to an automorphism $\tilde{\psi}$ of $H(\Sigma; L)$. This fact must be well-known, but we include a proof for completeness.

Choose arbitrary complements $H_\Sigma = A \oplus D$ and $H(\Sigma; L) = H_\Sigma \oplus B$ so that $H(\Sigma; L) = A \oplus (D \oplus B)$ is an orthogonal decomposition into symplectic subspaces A and $D \oplus B$. Note the symplectic isomorphism $H \cong H^*$ restricts to an isomorphism $D^* \cong B$. Consider the quotient $H_\Sigma/D \cong A$; the composition $\text{Sp}(H(\Sigma; L), D) \rightarrow \text{Sp}(H_\Sigma, D) \rightarrow \text{Sp}(A)$ admits an obvious section $\text{Sp}(A) \rightarrow \text{Sp}(H(\Sigma; L), D)$ by extending by the identity on $D \oplus B$. We may thus assume that the automorphism $\psi \in \text{Sp}(H_\Sigma, D)$ induces the identity on A . There is then some $f \in \text{Hom}(A, D)$ so that $\psi(a) = a + f(a)$ for all $a \in A$. Consider the transpose $f^\top \in \text{Hom}(D^*, A^*) \cong \text{Hom}(B, A)$. Extend ψ to B by $\tilde{\psi}(b) = b - f^\top(b)$ for all $b \in B$. To see that $\tilde{\psi}$ is an automorphism of H , it suffices to check that it preserves the pairing for elements $b \in B$ and $v \in H_\Sigma$. For $b \in B$ and $d \in D$, we have

$$(\tilde{\psi}(b), \tilde{\psi}(d)) = (b - f^\top(b), d) = (b, d) - (f^\top(b), d) = (b, d)$$

since $f^\top(b) \in A \subset D^\perp$. For $b \in B$ and $a \in A$, for which $(b, a) = 0$, we have

$$(\tilde{\psi}(b), \tilde{\psi}(a)) = (b - f^\top(b), a + f(a)) = (b, f(a)) - (f^\top(b), a) = 0$$

by the definition of f^\top . Thus $\tilde{\psi}$ is an automorphism of $H(\Sigma; L)$, as desired. \square

7.3 Orbits of nonseparating twists

Let $\Sigma = S = S_{g,1}$, so that $H(S) = H_1(S)$. We first consider nonseparating curves in S , which are always considered to be oriented. It was known to Johnson [5] that two nonseparating curves are in the same $\mathcal{I}(S)$ -orbit if and only if they are homologous. We thus consider two homologous nonseparating curves A_1 and A_2 with homology class $a \in H(S)$, and ask when they are in the same $\mathcal{K}(S)$ -orbit.

Note that a is nonzero, and is primitive in $H(S)$; $a^\perp \leq H(S)$ is the subspace

$$\{x \in H(S) \mid (a, x) = 0\}.$$

By $a^\perp \wedge a^\perp$ we mean the subspace of $H(S) \wedge H(S)$ spanned by $x \wedge y$ for $x, y \in a^\perp$, and by $a \wedge a^\perp$ we mean the subspace spanned by $a \wedge y$ for $y \in a^\perp$. Since $(a, a) = 0$, $a \wedge a^\perp$ is contained in $a^\perp \wedge a^\perp$. Finally, we say that a simple loop based at $*$ $\in \partial S$ is *clockwise-oriented* if upon identifying a neighborhood of $*$ with the upper half plane, the tail of the loop (leaving $*$) is to the left of the head of the loop (returning to $*$). The motivation for this condition is that any two nonseparating clockwise-oriented simple based loops lie in the same orbit under the mapping class group; without this condition, it is clear that a clockwise-oriented loop is never equivalent to a counterclockwise-oriented loop. Let $\pi = \pi_1(S_{g,1}, *)$ and let $\Gamma_j = \Gamma_j^T(\Sigma)$ be its lower central series.

Definition 7.4. Given two homologous nonseparating curves A_1 and A_2 with homology class $a \in H(S)$, we define $d_a(A_1, A_2) \in (a^\perp \wedge a^\perp)/(a \wedge a^\perp)$ as follows.

Represent A_1 and A_2 by simple based clockwise-oriented loops α_1, α_2 . Since $[\alpha_1] = [\alpha_2] = a$ in $H(S) = \Gamma_1/\Gamma_2$, it follows that $\alpha_2\alpha_1^{-1}$ lies in Γ_2 . Consider its class

$$t_a(\alpha_1, \alpha_2) := [\alpha_2\alpha_1^{-1}] \in \Gamma_2/\Gamma_3 = N(S) \simeq H(S) \wedge H(S).$$

We will see below that $t_a(A_1, A_2)$ is always contained in $a^\perp \wedge a^\perp \leq H(S) \wedge H(S)$. We define $d_a(A_1, A_2)$ to be the projection of $t_a(A_1, A_2)$ to $(a^\perp \wedge a^\perp)/(a \wedge a^\perp)$.

Remark 7.5. The loop α_2 representing A_2 is not uniquely defined, but any other representative α'_2 satisfies $\alpha'_2 = \xi\alpha_2\xi^{-1}$ in π , where $(a, [\xi]) = 0$. The difference between $t(\alpha_1, \alpha_2)$ and $t(\alpha_1, \alpha'_2)$ is thus equal (modulo Γ_3^T) to $[\alpha_2, \xi]$, which corresponds to $a \wedge [\xi]$ under the identification $\Gamma_2/\Gamma_3 \cong H \wedge H$. The same argument applies to α_1 . Since $a \wedge [\xi] \in a \wedge a^\perp$, it follows that $d_a(A_1, A_2) \in (a^\perp \wedge a^\perp)/(a \wedge a^\perp)$ is well-defined regardless of the choice of representatives.

Theorem 7.6. A_1 and A_2 lie in the same $\mathcal{K}(S)$ -orbit if and only if $d_a(A_1, A_2) = 0$.

Proof. Our goal is to prove that $d_a(A_1, A_2)$ as defined in Definition 7.4 agrees with the abstract invariant defined in (18). We first compute $\tau_S(\text{Stab}(\alpha))$. Fix a homology class b with $(a, b) = 1$, and note that

$$W_S = \bigwedge^3 H = \bigwedge^3 a^\perp \oplus b \otimes \bigwedge^2 a^\perp.$$

Let $\tilde{\Sigma} = (S_{g-1,3}, \tilde{\mathcal{P}}, *)$ where $\tilde{\mathcal{P}}$ is of the form $\{\{z_0\}, \{a_1, a_2\}\}$ and $*$ $\in z_0$. Note that $\tilde{\Sigma} = S_{g,1} = S$, and any mapping class stabilizing α lifts (non-uniquely) to $\tilde{\Sigma}$. Conversely, extension by the identity gives a surjection $\text{Mod}(\tilde{\Sigma}) \rightarrow \text{Stab}(\alpha)$; by Paris–Rolfen [9, Theorem 4.1(iii)], the kernel of this surjection is cyclic, generated by $T_{a_1}T_{a_2}^{-1}$. Let z_1 be the boundary component of $\tilde{\Sigma}$ corresponding to $\tilde{P}_1 = \{a_1, a_2\}$. By Theorem 5.7, noting that $D(\tilde{\Sigma}) = \langle a \rangle$, we have

$$0 \rightarrow \bigwedge^3 a^\perp \rightarrow W_{\tilde{\Sigma}} \rightarrow a^\perp \rightarrow 0$$

Since $\text{Mod}(\widetilde{\Sigma}) \rightarrow \text{Stab}(\alpha)$ is surjective, Theorem 5.12 implies that $\tau_S(\text{Stab}(\alpha))$ is the image of the map $\iota_*: W_{\widetilde{\Sigma}} \rightarrow W_S$ defined by $\delta_1(f) \wedge z_1 \mapsto 0$. But condition (II) of Definition 5.6 implies $f(a) = \delta_1(f) \wedge a$; it follows that the kernel of ι_* is cyclic, spanned by $a \wedge z_1$. (Note that $\tau_{\widetilde{\Sigma}}(T_{a_1}T_{a_2}^{-2}) = a \wedge z_1$ by Proposition 6.3, so we knew this must lie in $\ker \iota_*$.) The first factor $\bigwedge^3 a^\perp < W_{\widetilde{\Sigma}}$ is mapped isomorphically to $\bigwedge^3 a^\perp$; the other factor a^\perp/a is embedded as $b \otimes \delta_1(f) \wedge a$ in $b \otimes \bigwedge^2 a^\perp$. In particular, we have an isomorphism

$$W_S/\tau_S(\text{Stab}(\alpha)) \simeq a^\perp \wedge a^\perp/a \wedge a^\perp$$

defined by sending

$$f \in W_S \quad \mapsto \quad f(a) \in a^\perp \wedge a^\perp/a \wedge a^\perp. \quad (19)$$

At this point we need the following lemma: we can find $\varphi \in \mathcal{I}(\Sigma)$ so that $\varphi(\alpha_1) = \alpha_2$. For unbased loops this was known to Johnson [5], who proved that for nonseparating curves A_1 and A_2 , there exists $\varphi \in \mathcal{I}(\Sigma)$ with $\varphi(A_1) = A_2$ if and only if $[A_1] = [A_2]$. His argument extends to based loops without difficulty, as we now sketch. The classification of surfaces implies that we may find $\psi \in \text{Mod}(\Sigma)$ such that $\psi(\alpha_1) = \alpha_2$, as discussed before Definition 7.4. It suffices to find $\psi' \in \text{Stab}(\alpha_1)$ with $\Psi(\psi) = \Psi(\psi')$, for then $\psi \circ \psi'^{-1}$ is contained in $\mathcal{I}(\Sigma)$ and takes α_1 to α_2 . The fact that $\psi(\alpha_1) = \alpha_2$ implies that $\Psi(\psi)$ fixes $a \in H(\Sigma)$. But $\text{Stab}(\alpha_1)$ is the surjective image of the mapping class group $\text{Mod}(S - \alpha_1)$ of the cut surface $S - \alpha_1$. By Lemma 7.2 (or a slight variant), the image $\Psi(\text{Mod}(S - \alpha_1))$ is all of $\text{Sp}(H(S), a)$, so we may find some ψ' in $\text{Mod}(S - \alpha_1)$, and thus in $\text{Stab}(\alpha_1)$, with $\Psi(\psi') = \Psi(\psi)$ as desired.

Now choose $\varphi \in \mathcal{I}(\Sigma)$ so that $\varphi(\alpha_1) = \alpha_2$. The invariant $d_a(A_1, A_2)$ defined in (18) should by (19) be equal to $\tau_S(\varphi)(a)$. By Definition 5.2 this is the class of $\varphi(\alpha_1)\alpha_1^{-1} = \alpha_2\alpha_1^{-1}$, which coincides with $d_a(A_1, A_2)$ as defined in Definition 7.4. \square

7.4 Orbits of separating twists

We let $\Sigma = S = S_{g,1}$, and now consider separating curves in S . A separating curve C has a canonical orientation, by taking the boundary ∂S to be on the right side of C . The curve C separates S into two components S_1 and S_2 , where S_2 contains ∂S , and we define $V_C \leq H(S)$ to be the subspace of $H(S)$ spanned by $H_1(S_1)$; this induces an orthogonal splitting $H(S) = V_C \oplus V_C^\perp$. Johnson [5, Theorem 1A] proved that two separating curves C_1 and C_2 are in the same $\mathcal{I}(S)$ -orbit if and only if the subspaces V_{C_1} and V_{C_2} coincide. (Just as for nonseparating curves, this can be extended to simple based loops using Lemma 7.2.) We thus consider two separating curves C_1 and C_2 with $V_{C_1} = V_{C_2} = V$, and ask when they are in the same $\mathcal{K}(S)$ -orbit. Recall that $\omega_V \in \bigwedge^2 H(S)$ represents the restriction of the symplectic form ω to the symplectic subspace V .

Definition 7.7. Given a symplectic subspace $V \leq H$, let

$$Y_V := (V \otimes \bigwedge^2 V^\perp) \oplus (V^\perp \otimes \bigwedge^2 V).$$

Note that V^\perp embeds in the second factor of Y_V as $V^\perp \otimes \omega_V \leq V^\perp \otimes \bigwedge^2 V$; define

$$X_V := Y_V / (V^\perp \otimes \omega_V).$$

Given two separating curves C_1 and C_2 with $V_{C_1} = V_{C_2} = V$, we define $d_V(C_1, C_2) \in X_V$ as follows. Represent C_1 and C_2 by simple based loops γ_1, γ_2 . It is known that for a separating curve

we have $\gamma_i \in \Gamma_2$, and furthermore that $[\gamma_1] = [\gamma_2] = \omega_V$ under the identification $\Gamma_2/\Gamma_3 \cong \bigwedge^2 H(S)$. It follows that $\gamma_2 \gamma_1^{-1}$ lies in Γ_3 , and we consider its class

$$t_V(\gamma_1, \gamma_2) := [\gamma_2 \gamma_1^{-1}] \in \Gamma_3/\Gamma_4.$$

By Proposition 3.5, Γ_3/Γ_4 is isomorphic to $(H(S) \otimes \bigwedge^2 H(S))/(\bigwedge^3 H(S))$. Since Y_V is disjoint from $\bigwedge^3 H(S)$ as a subspace of $H(S) \otimes \bigwedge^2 H(S)$, this quotient restricts to an identification of Y_V with a subspace in Γ_3/Γ_4 . It turns out that $t(\gamma_1, \gamma_2)$ will lie inside this copy of Y_V , and we define $d_V(C_1, C_2)$ to be the projection of $t(\gamma_1, \gamma_2)$ to $X_V = Y_V/(V^\perp \otimes \omega_V)$.

Remark 7.8. The loop γ_2 representing C_2 is not uniquely defined, but any other representative γ'_2 satisfies $\gamma'_2 = \xi \gamma_2 \xi^{-1}$ in Γ , where $[\xi] \in V^\perp$. The difference between $t(\gamma_1, \gamma_2)$ and $t(\gamma_1, \gamma'_2)$ is equal (modulo Γ_4) to $[\gamma_2, \xi]$, which corresponds to $-[\xi] \otimes [\gamma_2]$ under the identification of Y_V with a subspace of G_3/G_4 . We noted above that $[\gamma_2] = \omega_V$ in $G_2/G_3 \cong \bigwedge^2 H$, so $[\xi] \otimes [\gamma_2] \in V^\perp \otimes \omega_V$. The same argument will apply to γ_1 , and it follows that $d_V(A_1, A_2) \in Y_V/(V^\perp \otimes \omega_V)$ is well-defined regardless of the choice of representatives.

Theorem 7.9. C_1 and C_2 lie in the same $\mathcal{K}(S)$ -orbit if and only if $d_V(C_1, C_2) = 0$.

Proof. Our goal is to prove that $d_V(A_1, A_2)$ as defined in Definition 7.7 agrees with the abstract invariant defined in (18). Let C be a separating curve with $V_C = V$, and represent C by a simple based loop γ . We first compute $\tau_\Sigma(\text{Stab}_{\mathcal{I}(S)}(C))$. Let Σ_1 and Σ_2 be the components of $S - C$; the surface Σ_1 is $S_{k,1}$ for some $1 \leq k \leq g$, while the surface Σ_2 is $S_{g-k,2}$ with the totally separated partition. Since C induces a splitting of $H(S)$ preserved by $\text{Stab}_{\text{Mod}(S)}(C)$, we have a decomposition $\text{Stab}_{\mathcal{I}(S)}(C) \simeq \mathcal{I}(\Sigma_1) \times \mathcal{I}(\Sigma_2)$. Note that $H(\Sigma_1) = V$, while $H(\Sigma_2) = V^\perp$, so Theorem 5.7 implies that $\tau_{\Sigma_1}(\mathcal{I}(\Sigma_1)) \simeq \bigwedge^3 V$ and $\tau_{\Sigma_2}(\mathcal{I}(\Sigma_2)) \simeq \bigwedge^3 V^\perp \oplus V^\perp$. Passing to the ambient surface S , Theorem 5.12 tells us that

$$\tau_S(\mathcal{I}(\Sigma_1)) = \bigwedge^3 V \leq \bigwedge^3 H(S), \quad \tau_S(\mathcal{I}(\Sigma_2)) = \bigwedge^3 V^\perp \oplus V^\perp \wedge \omega_V \leq \bigwedge^3 H(S).$$

We conclude that:

$$W_S/\tau(\text{Stab}(C)) \simeq (V \otimes \bigwedge^2 V^\perp) \oplus (V^\perp \otimes \bigwedge^2 V) \quad / \quad (V^\perp \otimes \omega_V) \quad (20)$$

It remains to check that the isomorphism (20) is computed by considering $[\varphi(\gamma)\gamma^{-1}] \in \Gamma_3/\Gamma_4$.

We will prove below that the homomorphism $\mathcal{I}(S) \rightarrow \Gamma_3/\Gamma_4$ defined by $\varphi \mapsto [\varphi(\gamma)\gamma^{-1}]$ is equal to the composition:

$$\mathcal{I}(S) \xrightarrow{\tau_S} \text{Hom}(H(S), \bigwedge^2 H(S)) \xrightarrow{\cong} H(S) \otimes \bigwedge^2 H(S) \xrightarrow{\pi_V} V \otimes \bigwedge^2 H(S) \rightarrow \Gamma_3/\Gamma_4 \quad (21)$$

Here $\pi_V: H(S) \otimes \bigwedge^2 H(S) \rightarrow V \otimes \bigwedge^2 H(S)$ is the projection induced by applying the orthogonal decomposition $H(S) = V \oplus V^\perp$ to the first factor, and the last map $V \otimes \bigwedge^2 H(S) \rightarrow \Gamma_3/\Gamma_4$ is the restriction of the bracket $\mathcal{L}_1 \otimes \mathcal{L}_2 \rightarrow \mathcal{L}_3$. Note that

$$V \otimes \bigwedge^2 H(S) = (\bigwedge^3 V) \oplus (V \otimes \bigwedge^2 V^\perp) \oplus (V^\perp \otimes \bigwedge^2 V).$$

The bracket has kernel $\bigwedge^3 H(S)$ by Proposition 3.5, which intersects $V \otimes \bigwedge^2 H(S)$ just in $\bigwedge^3 V$, so the image of the composition (21) is $Y_V = (V \otimes \bigwedge^2 V^\perp) \oplus (V^\perp \otimes \bigwedge^2 V)$ as claimed.

We now verify that the composition (21) is computed by $[\varphi(\gamma)\gamma^{-1}]$. Write $\gamma = [\alpha_1, \beta_1] \cdots [\alpha_k, \beta_k]$; there is some symplectic basis $\{a_i, b_i\}_{1 \leq i \leq g}$ of H so that the classes $\{[\alpha_i] = a_i, [\beta_i] = b_i\}_{1 \leq i \leq k}$ span

V. We will compute $\varphi(\gamma)\gamma^{-1}$ just as in the proof of Theorem 5.9. Set $\eta_\varphi(x) = x^{-1}\varphi(x)$ and note that $\eta_\varphi(x) \in \Gamma_2$ for all $x \in \Gamma$.

$$\begin{aligned}\varphi(\gamma) &= \varphi\left(\prod_{i=1}^k [\alpha_i, \beta_i]\right) = \prod_{i=1}^k [\varphi(\alpha_i), \varphi(\beta_i)] \\ &= \prod_{i=1}^k [\alpha_i \eta_\varphi(\alpha_i), \beta_i \eta_\varphi(\beta_i)] \\ &\equiv \prod_{i=1}^k [\alpha_i, \beta_i] [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i] \pmod{\Gamma_4} \\ &\equiv \left(\prod_{i=1}^k [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i]\right) \cdot \gamma \pmod{\Gamma_4}\end{aligned}$$

Thus $[\varphi(\gamma)\gamma^{-1}] \in \Gamma_3/\Gamma_4$ is represented by $\prod_{i=1}^k [\alpha_i, \eta_\varphi(\beta_i)] [\eta_\varphi(\alpha_i), \beta_i]$. Since $\eta_\varphi(\alpha_i)$ and $\eta_\varphi(\beta_i) \in \Gamma_2$ represent $\tau_S(\varphi)(a_i)$ and $\tau_S(\varphi)(b_i) \in \bigwedge^2 H$, we see that $[\varphi(\gamma)\gamma^{-1}] \in \Gamma_3/\Gamma_4$ is the image of

$$\sum_{i=1}^k a_i \otimes \tau(f)(b_i) - b_i \otimes \tau(f)(a_i) \in V \otimes \bigwedge^2 H(S) \quad (22)$$

under the bracket map. As in the proof of Theorem 5.9 the element (22) represents $\pi_V(\tau_S(\varphi))$, verifying the claim. \square

Remark 7.10. In light of this theorem, our proof in Theorem 5.9 that $\tau_S(\varphi) \in \bigwedge^3 H(S)$ can be viewed as follows. Let C be a separating curve of genus g , so that $V = H(S)$. By Theorem 7.9, $\varphi(C)$ and C lie in the same $\mathcal{K}(S)$ -orbit if and only if $\tau_S(\varphi)$ vanishes in Γ_3/Γ_4 , i.e. vanishes modulo $\bigwedge^3 H(S)$. But since there is only one genus- g separating curve on S , we necessarily have $\varphi(C) = C$ for any $\varphi \in \mathcal{I}(S)$, and so we conclude that $\tau_S(\varphi) \in \bigwedge^3 H(S)$ for all $\varphi \in \mathcal{I}(S)$.

Remark 7.11. We remark that more care must be taken when considering orbits of separating curves and multicurves when the partition on Σ is not totally separated. Consider the lantern $S = S_{0,4}$ depicted in Figure 6.3, but with partition given by $\mathcal{P} = \{\{A_0, A_2\}, \{A_1, A_3\}\}$. Here A_1, A_2, A_3 are the three boundary components in the center and A_0 is the outer boundary component. The two curves α and β depicted in Figure 6.3 are both \mathcal{P} -separating. Note that the complementary components U_1 and U_2 of $S - \alpha$ determine the subspaces $\langle a_1 = -a_3 \rangle$ and $\langle a_2 = -a_0 \rangle$ respectively, and the complementary components V_1 and V_2 of $S - \beta$ determine the same subspaces.

Nevertheless α and β do not lie in the same $\mathcal{I}(\Sigma)$ -orbit, as can be seen by considering the whole surface $\widehat{\Sigma}$. The complementary components of $\widehat{\Sigma} - \beta$ determine $\langle a_1, b_3 - b_1 \rangle$ and $\langle a_2, b_2 \rangle$, while the complementary components of $\widehat{\Sigma} - \alpha$ determine subspaces $\langle a_1, b_3 - b_1 - a_2 \rangle$ and $\langle a_2, b_2 + a_1 \rangle$. This shows there is no element of $\mathcal{I}(\widehat{\Sigma})$ taking α to β , and thus certainly no element of $\mathcal{I}(\Sigma)$ taking α to β .

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